Better use of jets shapes

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N-subjettiness

$$\tau_{21} = \frac{\tau_2^{(\beta)}(\text{jet; axes})}{\tau_1^{(\beta)}(\text{jet; axes})} = \frac{\sum_{i\in\text{constits}} z_i \min(\theta^\beta_{i,a_{2,1}}, \theta^\beta_{i,a_{2,2}})}{\sum_{i\in\text{constits}} z_i \, \theta^\beta_{i,a_{1,1}}}$$

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Parameters:

- β**:**
  - **give more or less weight to large/small angles**
  - $\beta \sim 2$ **seems slightly preferred in MC simulations**
  - $\beta \sim 1$ **should be less sensitive to non-perturbative effects and PU**

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  - **optimal, declustering, winner-takes-all, ...**
  - **For a given** β**, generalised-**k_t**(**$p = 1/\beta$**)**$\sim$**optimal**
  - **use WTA for** $\beta \leq 1$

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  - **apply on full jet? (more discrimination, more NP Sensitive)**
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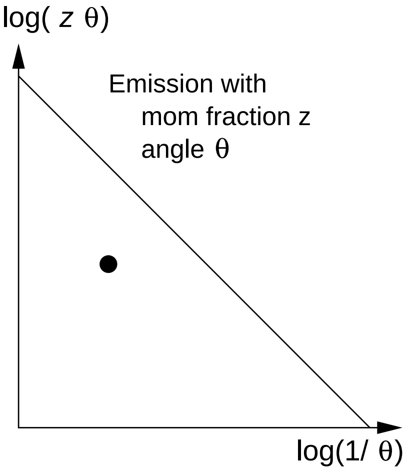
Parameters:

- β: **focus on** $\beta = 2$
  - **give more or less weight to large/small angles**
  - $\beta \sim 2$ **seems slightly preferred in MC simulations**
  - $\beta \sim 1$ **should be less sensitive to non-perturbative effects and PU**
- **choice of axes: focus on gen-k_t(1/2) (or optimal)**
  - **optimal, declustering, winner-takes-all, ...**
  - **For a given β, generalised-k_t($p = 1/\beta$)$\sim$optimal**
  - **use WTA for $\beta \leq 1$**
- **choice of jet: study several options**
  - **What to do with soft-and-large-angle emissions?**
  - **apply on full jet? (more discrimination, more NP Sensitive)**
  - **apply on groomed jet? (less discrimination, less NP Sensitive)**

Concepts

in cartons

Anatomy of the phase-space



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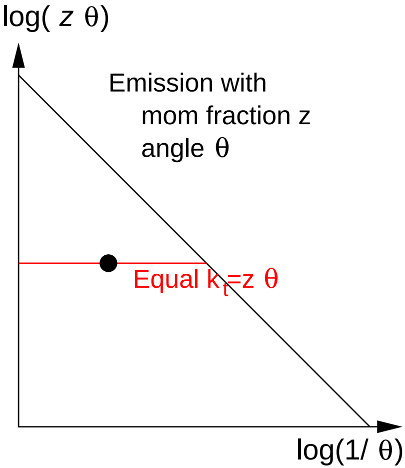

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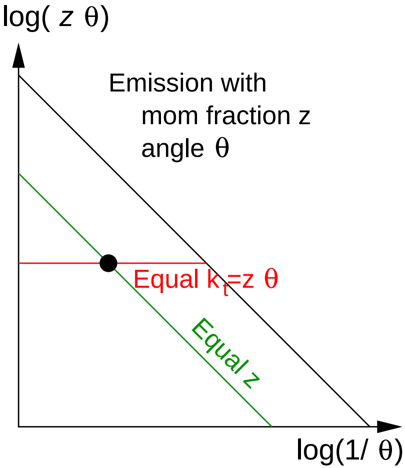

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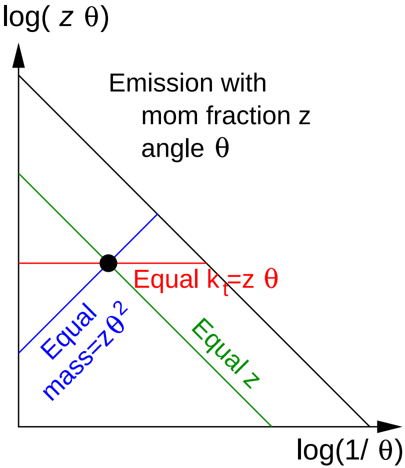

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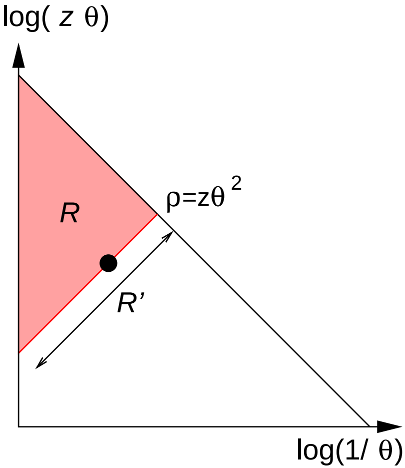


Example: Jet mass

$$\frac{\rho}{\sigma}\frac{d\sigma}{d\rho} = R'_m \exp(-R_m)$$

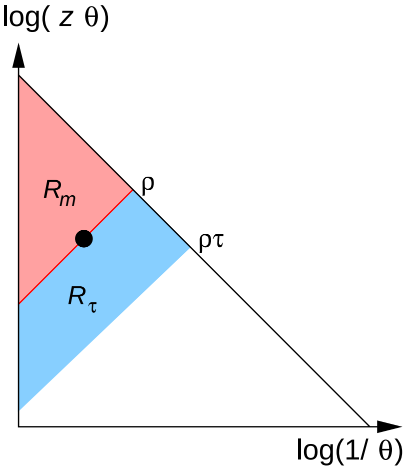
veto on larger-mass (Sudakov)

$R_m \sim \frac{\alpha_s C_R}{2\pi} \log^2(1/\rho)$

emission of given mass

$R'_m \sim \frac{\alpha_s C_R}{\pi} \log(1/\rho)$

Anatomy of the phase-space



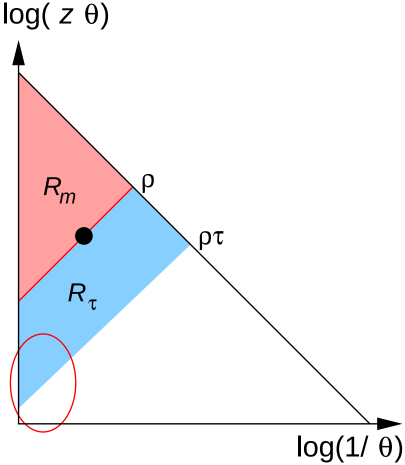
Jet mass with a cut on τ_{21}:

$$\frac{\rho}{\sigma}\frac{d\sigma}{d\rho}\bigg|_{<\tau} = R'_m \exp(-R_m - R_\tau)$$

Extra suppression

$$R_m \sim \frac{\alpha_s C_R}{2\pi}\log^2(1/\rho)$$

becomes $\quad R_m + R_\tau \sim \frac{\alpha_s C_R}{2\pi}\log^2(1/\tau\rho)$



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Soft-and-large-angle radiation:

- performance gain (R_τ)
- large NP effects

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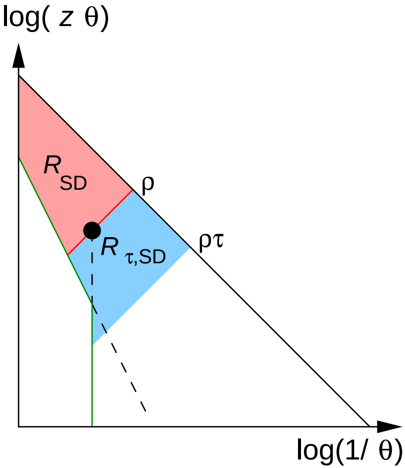


Jet mass with SoftDrop+a cut on τ_{21}:

$$\frac{\rho}{\sigma}\frac{d\sigma}{d\rho}\Big|_{\text{SD},<\tau} = R'_{\text{SD}} \exp(-R_{\text{SD}} - R_{\tau,\text{SD}})$$

Reduced NP sensitivity

But less performant:

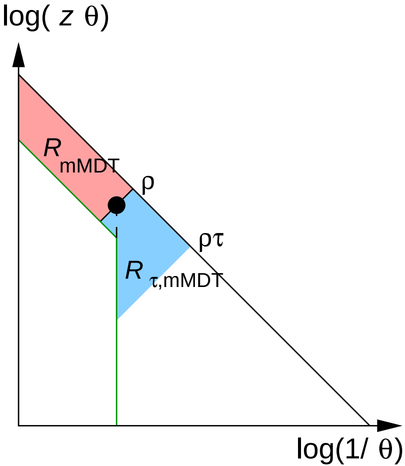
$$R_m + R_\tau \sim \frac{\alpha_s C_R}{2\pi} \log^2(1/\tau\rho)$$

becomes

$$R_{\text{SD}} + R_{\tau,\text{SD}} \sim \frac{\alpha_s C_R}{2\pi} \frac{\beta}{2+\beta} \log^2(1/\tau\rho)$$

but $R'_{\text{SD}} < R'_m$

Anatomy of the phase-space



mMDT($\equiv$SD($\beta = 0$))+a cut on τ_{21}:

$$\left.\frac{\rho}{\sigma}\frac{d\sigma}{d\rho}\right|_{\text{MD},<\tau} = R'_{\text{MD}} \exp(-R_{\text{MD}} - R_{\tau,\text{MD}})$$

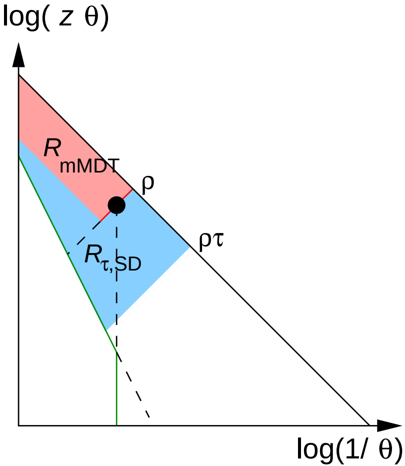
Efficient 2-prong tagger

Even less performant τ_{21} cut:

$$R_{\text{MD}} + R_\tau \sim$$
$$\sim \frac{\alpha_s C_R}{2\pi} \log(1/z_{cut}) \log(1/\tau\rho)$$

but $R'_{\text{MD}} < R'_{\text{SD}}$

Anatomy of the phase-space

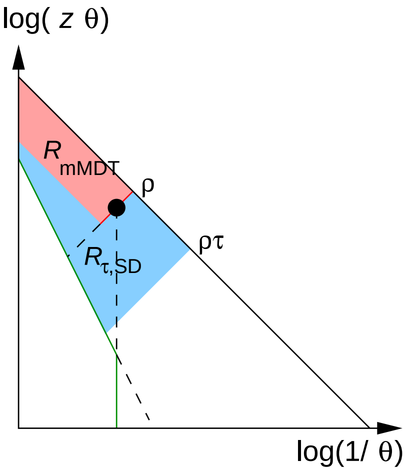


Combine the two effects:

- mMDT to get 2 prongs
  Use that for ρ and τ_1
- "Gentle" SD to reduce NP effect
  Use that for τ_2

$$\tau_{21} = \frac{\tau_2(\text{SD})}{\tau_1(\text{mMDT})}$$

Anatomy of the phase-space



Jet mass with "only" SD:

$$\frac{\rho}{\sigma}\frac{d\sigma}{d\rho}\bigg|_{\text{SD},<\tau} = R'_{\text{SD}}\,\exp(-R_{\text{SD}} - R_{\tau,\text{SD}})$$

becomes our "mixed" case:

$$\frac{\rho}{\sigma}\frac{d\sigma}{d\rho}\bigg|_{\text{mix},<\tau} = R'_{\text{MD}}\,\exp(-R_{\text{MD}} - R_{\tau,\text{SD}})$$

- Larger Sudakov comapred to MD
- Smaller pre-factor compared to SD

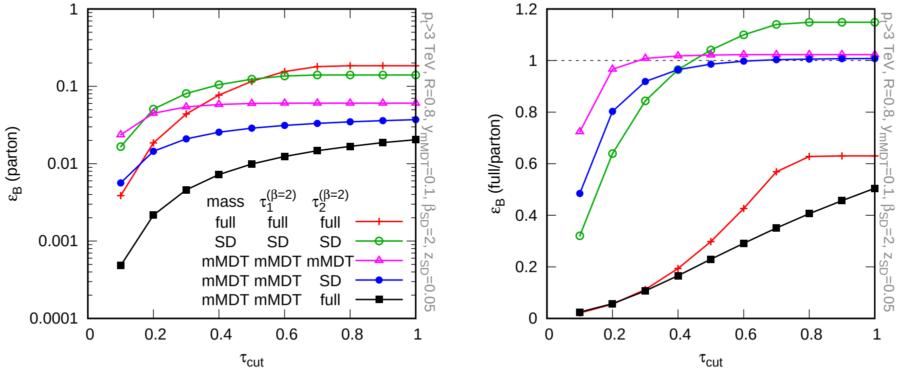
Gain in all cases

Monte-Carlo validation

Monte-Carlo validation

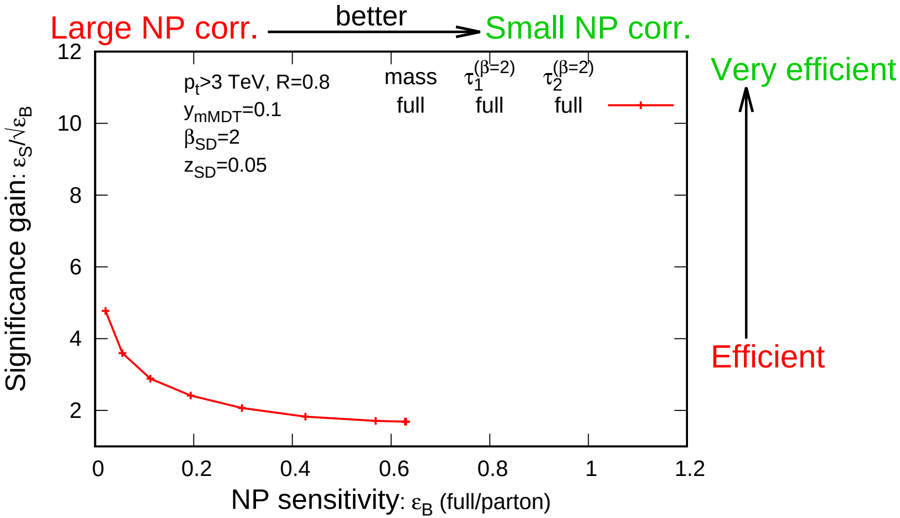
Background rate

NP effects



- "Mixed" case shows improvement
- Trade-off between performance and NP sensitivity

Monte-Carlo validation



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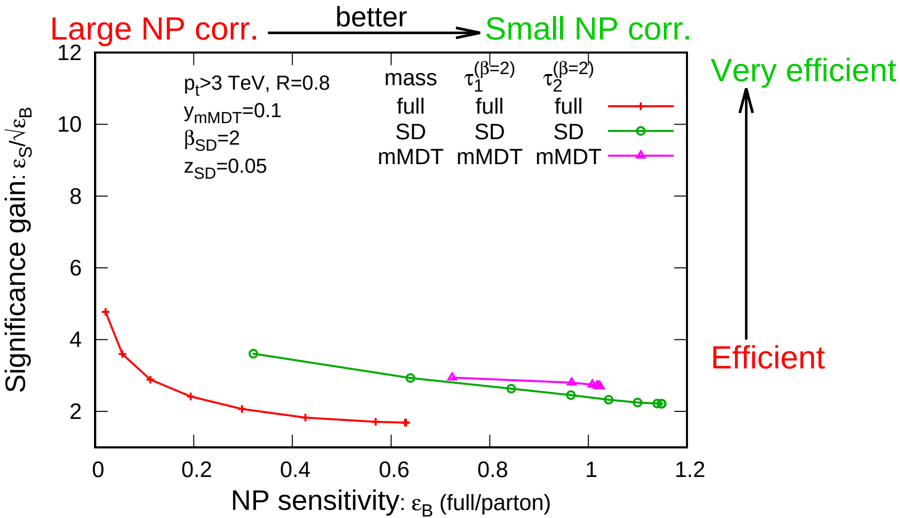

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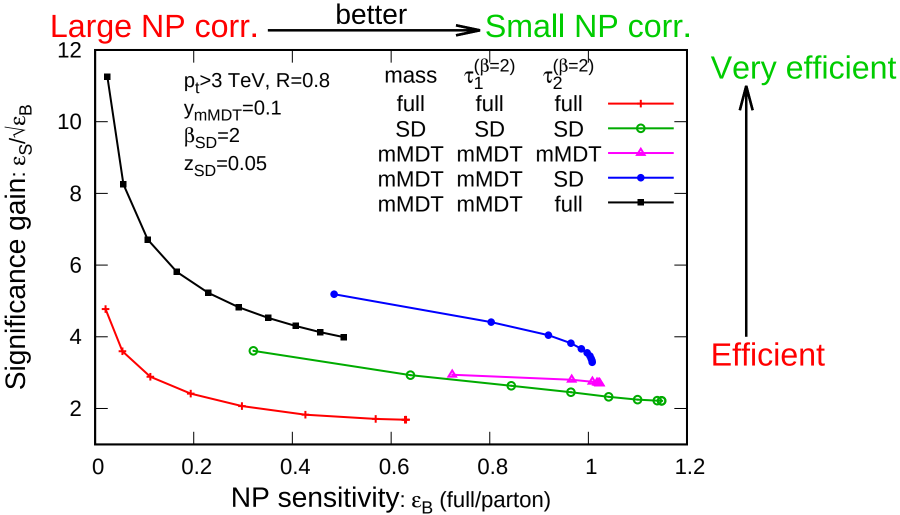

Towards a better analytic control

Calculation: idea

Target accuracy:

- $\tau \ll 1$: Include all double logs: $\alpha_s^n(\log^2(1/\tau), \log(1/\tau)\log(1/\rho))^n$
- τ finite: Include leading logs of ρ: $\alpha_s^n \log^n(1/\rho) f(\tau)$

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Details:

- Start with τ_{21} for the plain jet
- Consider n emissions $(z_1, \theta_1), \ldots, (z_n, \theta_n)$

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At this accuracy, we have (thanks to $\beta = 2$ and axes choice)

- $\rho = \tau_1 = \sum_{i=1}^{n} z_i \theta_i^2$
- $\tau_2 = \tau_1 - \max\{z_i \theta_i^2\}$ (min or gen-k_t axes)

Calculation: method

Start from multiple-emissions assuming "emission 1 most massive"

$$\frac{\rho\tau}{\sigma}\frac{d^2\sigma}{d\rho\, d\tau} = \lim_{\epsilon\to 0} \exp\left[-\int_{z\theta^2>\epsilon}^{1} \frac{d\theta^2}{\theta^2} P(z)dz \frac{\alpha_s(z\theta)}{2\pi}\right] \sum_{n=2}^{\infty} \frac{1}{(n-1)!}$$

$$\prod_{i=1}^{n} \int_{z_i\theta_i^2>\epsilon}^{1} \frac{d\theta_i^2}{\theta_i^2} P(z_i)dz_i \frac{\alpha_s(z_i\theta_i)}{2\pi} \prod_{i=2}^{n} \Theta(z_i\theta_i^2 < z_1\theta_1^2)$$

$$\rho\delta(\rho - \sum_{i=1}^{n} z_i\theta_i^2)\, \tau\delta(\tau - \sum_{i=2}^{n} z_i\theta_i^2 / \sum_{i=1}^{n} z_i\theta_1^2)$$

- Virtual corrections
- Real emissions phase-space (with "1" most massive)
- Constraints on mass and τ

Calculation: method

Only depends on $\rho_i = z_i\theta_i^2$ (thanks to $\beta = 2$)

$$\frac{\rho\tau}{\sigma}\frac{d^2\sigma}{d\rho\,d\tau} = \lim_{\epsilon\to 0} \exp\left[-\int_\epsilon^1 \frac{d\tilde{\rho}}{\tilde{\rho}} R'(\tilde{\rho})\right] \sum_{n=2}^{\infty} \frac{1}{(n-1)!}$$

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$$\rho\delta(\rho - \sum_{i=1}^{n} \rho_i)\, \tau\delta(\tau - \sum_{i=2}^{n} \rho_i / \sum_{i=1}^{n} \rho_1)$$

- $R'(\rho) = \int \frac{d\theta^2}{\theta^2} P(z) dz \frac{\alpha_s(z\theta)}{2\pi}\, \rho\delta(\rho - z\theta^2) \sim \alpha_s \log(1/\rho)$
- Easily written for SoftDrop (use $R = R_{SD}$ instead of $R = R_m$)

Calculation: method

After "CEASAR-like" manipulations:

$$\frac{\rho\tau}{\sigma}\frac{d^2\sigma}{d\rho\, d\tau} \overset{\tau<1/2}{=} \frac{e^{-R(\rho\tau)-\gamma_E R'(\rho\tau)}}{\Gamma(R'(\rho\tau))}\frac{R'((1-\tau)\rho)}{1-\tau}$$

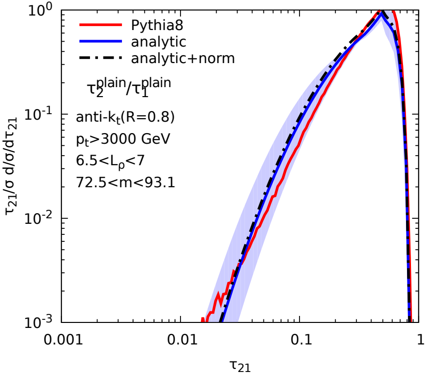
$$\overset{\tau>1/2}{=} \frac{e^{-R(\rho(1-\tau))-\gamma_E R'(\rho(1-\tau))}}{\Gamma(R'(\rho(1-\tau)))}\frac{R'((1-\tau)\rho)}{1-\tau} \times \frac{\tau}{1-\tau} f_{\rm ME}\left(\frac{\tau}{1-\tau}, R'(\rho(1-\tau))\right)$$

- Constraints give $\rho_1 = (1-\tau)\rho$, $\sum_{i=2}^n \rho_i = \rho\tau$
- $\tau = 1/2$ is the limit at which we need more than 2 emissions

with

$$\frac{e^{-\gamma_E R'}}{\Gamma(R')} f_{\rm ME}(x, R') = \lim_{\varepsilon\to 0}\sum_{n=1}^{\infty}\frac{R'^n}{n!}\prod_{i=1}^{n}\int_{\varepsilon}^{1}\frac{dx_i}{x_i}\, e^{-R'\log(1/\varepsilon)}\delta(x-\sum_{i=1}^{n}x_i),$$

Comparison with Monte-Carlo



- Uncertainty band: Vary between ρ, $\rho/2$ and 2ρ
- $1 + \mathcal{O}(\alpha_s)$ normalisation
- Small-τ behaviour well reproduced

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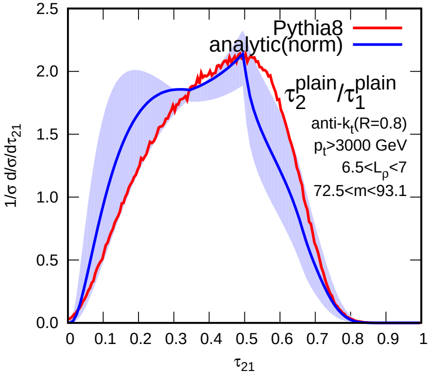

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  Vary between ρ, $\rho/2$ and 2ρ
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- Good overall description
- Kinks
  - 1/2: $\tau > \frac{1}{2}$ requires ≥ 3 emissions
    Smeared by subleading effects
  - 0.34: Start of 2ndary emissions
    Exact position subleading

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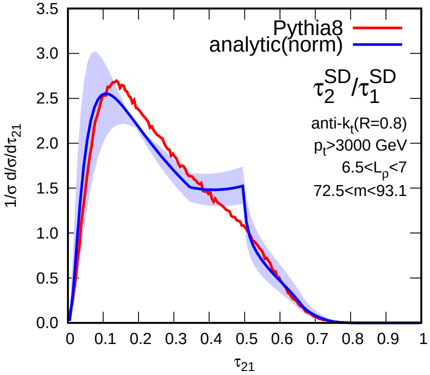

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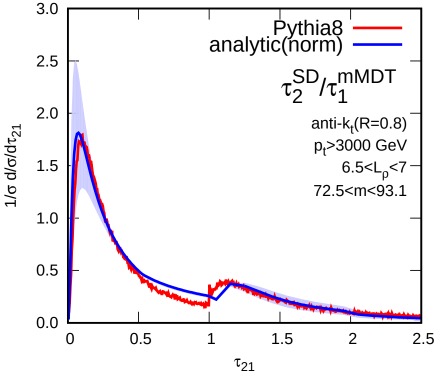

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  - 0.34: Start of 2ndary emissions
    Exact position subleading
- Works also with grooming
- Similar also for "mixed" (mMDT+SD)

More generic cases

- $\beta = 1$:
  - Two different orderings (mass and k_t)
  - but k_t ordering has simplifications
  - Preliminary results on the way
  - More complex: 3 “numerical” integrations instead of 1

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  - Interesting to compare with SCET results (also conceptually)
- τ_{32}: extends almost straightforwardly for $\beta = 2$

Summary

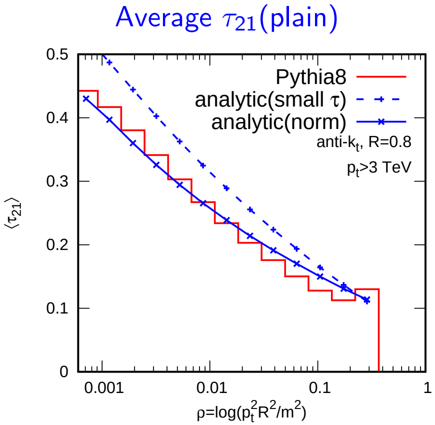
This talk

- Combination of tagging, shape constraint (and grooming) based on first-principle understanding
- generic method to compute jet shape distributions easier for $\tau_{21}^{(\beta=2)}$

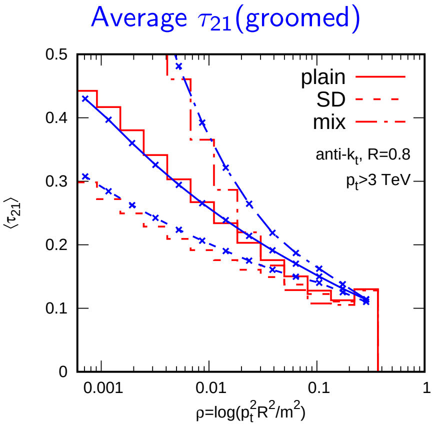
Future plans

- Finalise $\tau_{21}^{(\beta=1)}$ and D_2
- Match to fixed order (also for signal)
- Parameter optimisation based on analytic calcualtions

Average τ



Average τ_{21}(plain)
Pythia8
analytic(small τ)
analytic(norm)
anti-k$_t$, R=0.8
p$_t$>3 TeV
⟨τ$_{21}$⟩
ρ=log(p$_t^2$R^2/m^2)
0.5
0.4
0.3
0.2
0.1
0
0.001
0.01
0.1
1



Average τ_{21}(groomed)
plain
SD
mix
anti-k$_t$, R=0.8
p$_t$>3 TeV
⟨τ$_{21}$⟩
ρ=log(p$_t^2$R^2/m^2)
0.5
0.4
0.3
0.2
0.1
0
0.001
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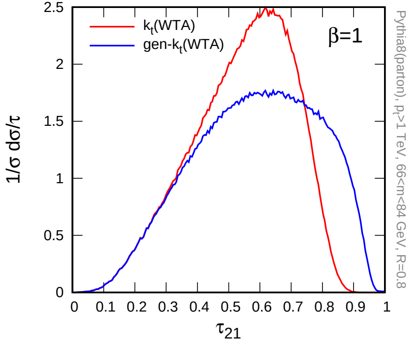
0.1
1

Axes dependence



Calculation: more details

Start from multiple-emissions assuming "emission 1 most massive"

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$$\prod_{i=1}^{n} \int_\epsilon^1 \frac{d\rho_i}{\rho_i} R'(\rho_i) \prod_{i=2}^{n} \Theta(\rho_i < \rho_1)$$

$$\rho\delta(\rho - \sum_{i=1}^{n} \rho_i)\, \tau\delta(\tau - \sum_{i=2}^{n} \rho_i / \sum_{i=1}^{n} \rho_1)$$

- $R'(\rho) = \int \frac{d\theta^2}{\theta^2} P(z) dz \frac{\alpha_s(z\theta)}{2\pi}\, \rho\delta(\rho - z\theta^2) \sim \alpha_s \log(1/\rho)$
- Easily written for SoftDrop (use $R = R_{SD}$ instead of $R = R_m$)

Calculation: more details

Use mass constraint to get rid of $i = 1$ integration (and rename emissions)

$$\frac{\rho\tau}{\sigma}\frac{d^2\sigma}{d\rho\, d\tau} = \lim_{\epsilon\to 0} \exp\left[-\int_\epsilon^1 \frac{d\tilde\rho}{\tilde\rho} R'(\tilde\rho)\right]\sum_{n=1}^{\infty}\frac{1}{n!}$$

$$\prod_{i=1}^{n}\int_\epsilon^1 \frac{d\rho_i}{\rho_i} R'(\rho_i)\prod_{i=2}^{n}\Theta(\rho_i < (1-\tau)\rho)$$

$$\frac{R'((1-\tau)\rho)}{1-\tau}\,\rho\tau\delta(\rho\tau - \sum_{i=1}^{n}\rho_i)$$

Have to consider 2 cases:

- $\tau < 1/2$: The constraint $\rho\tau = \sum_i \rho_i$ implied $\rho_i < (1-\tau)\rho$
- $\tau > 1/2$: The upper bound on ρ_i is set by $\rho_i < (1-\tau)\rho$

Calculation: more details

For $\tau < 1/2$, rescale all emissions by $\rho\tau$

$$\frac{\rho\tau}{\sigma}\frac{d^2\sigma}{d\rho\, d\tau} = \lim_{\epsilon\to 0} \exp\left[-R(\rho\tau) - \int_\epsilon^1 \frac{d\tilde{\rho}}{\tilde{\rho}} R'(\rho\tau)\right] \sum_{n=1}^{\infty}\frac{1}{n!}$$

$$\prod_{i=1}^{n}\int_\epsilon^1 \frac{d\zeta_i}{\zeta_i}[R'(\rho\tau)]^n$$

$$\frac{R'((1-\tau)\rho)}{1-\tau}\,\delta(1-\sum_{i=1}^{n}\zeta_i)$$

Notes:

- We have defined $\zeta_i = \rho_i/(\rho\tau)$
- We have replaced $\epsilon \to \rho\tau\epsilon$
- After factoring out $\exp[-R(\rho\tau)]$, all R' can be taken at $\rho\tau$.
- R and R' should include a C_A term ($\sim \alpha_s \log^2((1-\tau)/\tau)$)

For $\tau > 1/2$, rescale all emissions by $\rho(1-\tau)$

$$\frac{\rho\tau}{\sigma}\frac{d^2\sigma}{d\rho\, d\tau} = \lim_{\epsilon\to 0} \exp\left[-R(\rho(1-\tau)) - \int_\epsilon^1 \frac{d\tilde\rho}{\tilde\rho} R'(\rho(1-\tau))\right] \sum_{n=1}^{\infty} \frac{1}{n!}$$

$$\prod_{i=1}^{n} \int_\epsilon^1 \frac{d\zeta_i}{\zeta_i} \left[R'(\rho(1-\tau))\right]^n$$

$$\frac{R'((1-\tau)\rho)}{1-\tau}\,\frac{\tau}{1-\tau}\delta\Big(\frac{\tau}{1-\tau} - \sum_{i=1}^{n}\zeta_i\Big)$$

Notes:

- We have defined $\zeta_i = \rho_i/(\rho(1-\tau))$
- We have replaced $\epsilon \to \rho(1-\tau)\epsilon$
- After factoring out $\exp[-R(\rho(1-\tau))]$, all R' can be taken at $\rho(1-\tau)$.

Calculation: more details

In the end, we get

$$\frac{\rho\tau}{\sigma}\frac{d^2\sigma}{d\rho\, d\tau} \overset{\tau<1/2}{=} \frac{e^{-R(\rho\tau)-\gamma_E R'(\rho\tau)}}{\Gamma(R'(\rho\tau))}\frac{R'((1-\tau)\rho)}{1-\tau}$$

$$\overset{\tau>1/2}{=} \frac{e^{-R(\rho(1-\tau))-\gamma_E R'(\rho(1-\tau))}}{\Gamma(R'(\rho(1-\tau)))}\frac{R'((1-\tau)\rho)}{1-\tau}$$

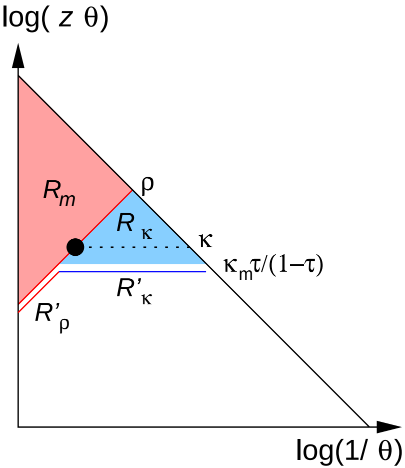
$$\times \frac{\tau}{1-\tau} f_{\rm ME}\left(\frac{\tau}{1-\tau}, R'(\rho(1-\tau))\right)$$

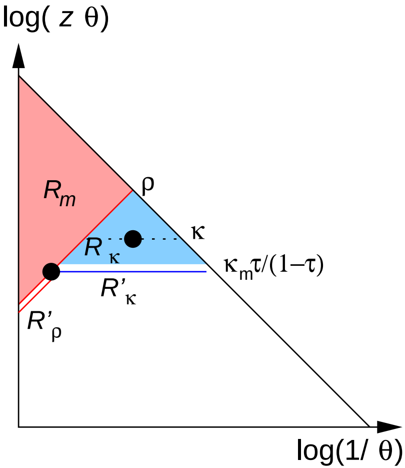
We have defined

$$\frac{e^{-\gamma_E R'}}{\Gamma(R')} f_{\rm ME}(x, R') = \lim_{\varepsilon\to 0}\sum_{n=1}^{\infty}\frac{R'^n}{n!}\prod_{i=1}^{n}\int_\varepsilon^1 \frac{dx_i}{x_i}\, e^{-R'\log(1/\varepsilon)}\delta(x-\sum_{i=1}^{n} x_i),$$

which can be rewritten as an inverse Laplace transform.

$\tau_{21}^{(\beta=1)}$ (preliminary)





$\tau_{21}^{(\beta=1)}$ (preliminary)

$$\frac{\rho}{\sigma}\frac{d^2\sigma}{d\rho\, d\tau} = \int_\rho^{\sqrt{\rho}} \frac{d\kappa_m}{\kappa_m}\frac{2\alpha_s(\kappa_m)C_R}{\pi}\frac{e^{-R_\rho(\rho)-\gamma_E R'_\rho(\tilde\kappa)-R_\kappa(\tilde\kappa)-\gamma_E R'_\kappa(\tilde\kappa)}}{\Gamma(1+R'_\rho(\tilde\kappa))\Gamma(R'_\kappa(\tilde\kappa))}$$

$$\frac{\kappa_m}{(1-\tau)^2\tilde\kappa} f_{ME}\left(\frac{\tau}{1-\tau}\frac{\kappa_p}{\tilde\kappa}; R'_\kappa(\tilde\kappa)\right)$$

$$+\int_\rho^{\sqrt{\rho}} \frac{d\kappa_m}{\kappa_m}\frac{2\alpha_s(\kappa_m)C_R}{\pi}\int_{\frac{1-\tilde t}{\tilde t}}^{\sqrt{\rho}} \frac{d\kappa_p}{\kappa_p} R'_\kappa(\kappa_p;\rho)e^{-R(\rho)}$$

$$\frac{e^{-\gamma_E R'_\rho(\bar\kappa)-R_\kappa(\bar\kappa)-\gamma_E R'_\kappa(\bar\kappa)}}{\Gamma(1+R'_\rho(\bar\kappa))\Gamma(R'_\kappa(\bar\kappa))}\frac{\kappa_p}{(1-\tau)^2\bar\kappa} f_{ME}\left(\frac{\tau}{1-\tau}\frac{\kappa_p}{\bar\kappa}-\frac{\kappa_m}{\bar\kappa}; R'_\kappa(\bar\kappa)\right)$$

with

$$\tilde\tau = \min(\tau, 1/2), \quad \tilde\kappa = \frac{\tilde\tau}{1-\tilde\tau}\kappa_m, \quad \bar\kappa = \min\left(\frac{\tau}{1-\tau}\kappa_p - \kappa_m, \kappa_p\right)$$

$$R_\rho(\kappa) = \int_\rho^\kappa \frac{d\kappa'}{\kappa'}\frac{2\alpha_s(\kappa')C_R}{\pi}$$

$$R_\kappa(\kappa) = \int_{\rho/\kappa}^\kappa \frac{d\theta}{\theta}\frac{2\alpha_s(\kappa)C_R}{\pi}$$