Towards a predictive high energy evolution

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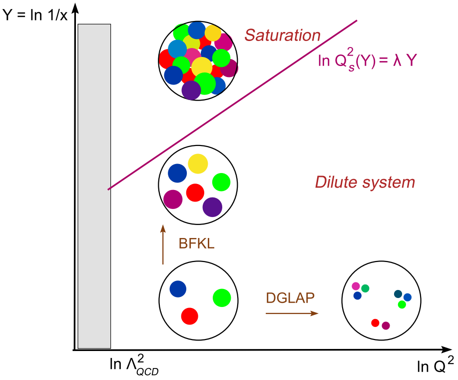
GDR QCD — Partons and Nuclei
June 1-2 2017

with Edmond Iancu, Jose Madrigal, Al Mueller and Dionysis Triantafyllopoulos

Evolution towards high energy



- 2 scales:
  - size r ($\sim 1/Q$)
  - energy ($\sim 1/x$)
- 2 evolutions
  - DGLAP: small r
  - BFKL/BK/JIMWLK/ saturation: high energy

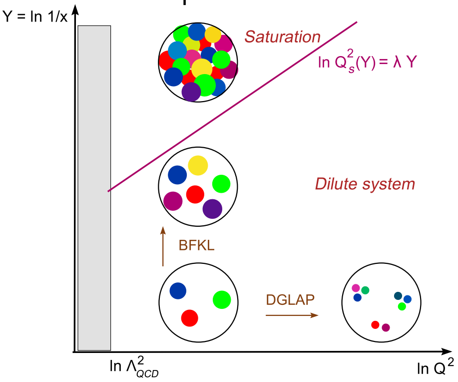
This talk

Many features qualitatively described by saturation physics

How **quantitative** can we make that statement?

Evolution towards high energy

Standard picture for "partons in the proton"



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This talk

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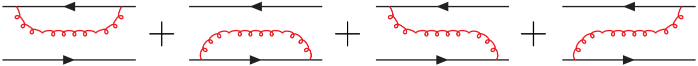
How **quantitative** can we make that statement?

Do (as much as you can) from **first-principles QCD**

Balitsky-Kovchegov equation (LO)



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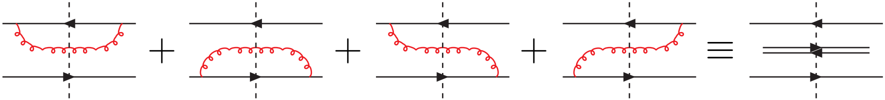

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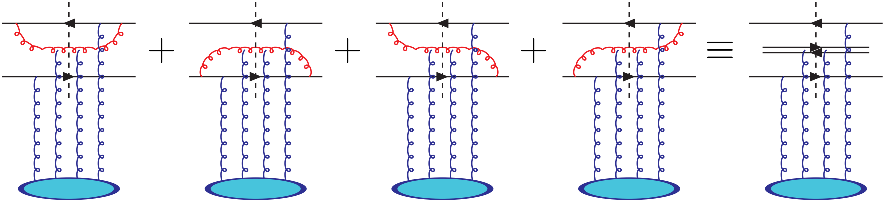

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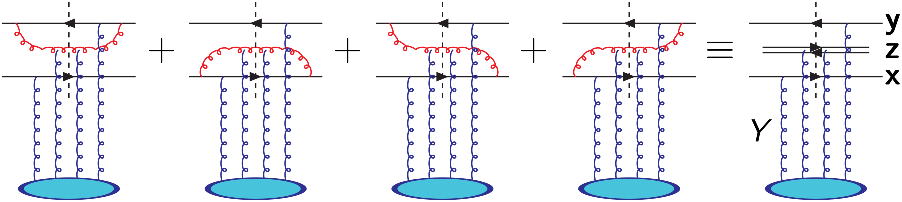
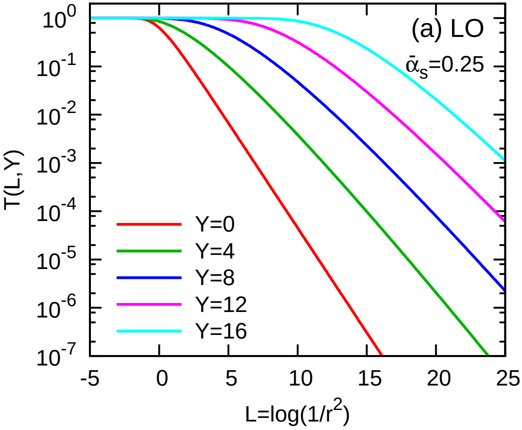


$$\partial_Y S(\mathbf{x},\mathbf{y};Y) = \int d^2\mathbf{z}\,\frac{\bar{\alpha}_s}{2\pi}\mathcal{M}_{\mathbf{xyz}}\left[S(\mathbf{x},\mathbf{z};Y)S(\mathbf{z},\mathbf{y};Y) - S(\mathbf{x},\mathbf{y};Y)\right]$$

$$\mathcal{M}_{\mathbf{xyz}} = \frac{(\mathbf{x}-\mathbf{y})^2}{(\mathbf{x}-\mathbf{z})^2(\mathbf{z}-\mathbf{y})^2}$$

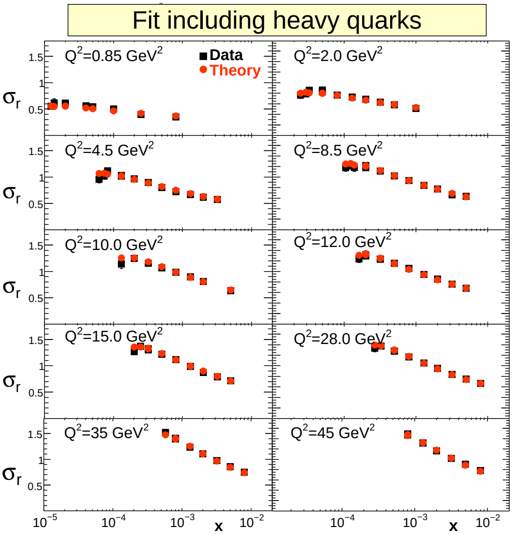
LO evolution of the dipole amplitude

Work with $T = 1 - S$



- Saturation for large dipoles
- "dilute" tail for small dipoles
- saturation scale $Q_s(Y)$ increasing with Y

Description of the HERA data



[AAMQS, 10]

- include running-coupling
- (several) Fits of S to HERA data
- reasonable description
- predictions for diffraction
- predictions for (fwd) particle production in pA
- not easy to include heavy flavour
- parameters not always natural

BK evolution at NLO

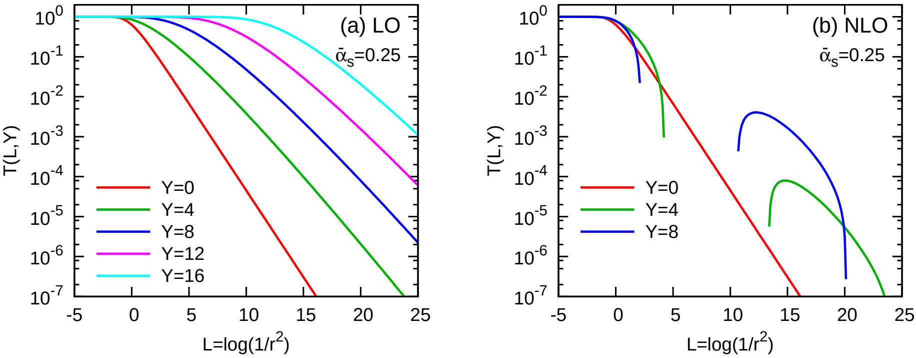
[Balitsky,Chirilli, 2008; Kovner, Lublinski, Mullian, 2013]

Compute next-to-leading-log(Y) corrections to the evolution:

$$\partial_Y S_{\mathbf{xy}}(Y) = \frac{\bar{\alpha}_s}{2\pi}\mathcal{M}\otimes S(Y) + \left(\frac{\bar{\alpha}_s}{2\pi}\right)^2 \mathcal{M}_{NLO}\otimes S(Y)$$

- LO: $\bar{\alpha}_s^n \log^n(Y)$ (leading log)
- NLO: $\bar{\alpha}_s^{n+1} \log^n(Y)$ (next-to-leading log)
- includes
  - terms of the same form as LO
  - terms involving 2-gluon emissions
  - running-coupling corrections

Instabilities of NLO BK



- Clear instability (T goes negative)
- See also work by T.Lappi and H.Mäntysaari
- Known issue in BFKL as well

[Ciafaloni,Colferai,Salam,Statso; Altarelli,Forte,Ball]

A problematic term

Go back to the BK equation at NLO:

$$\partial_Y S_{\mathbf{xy}}(Y) = \frac{\bar{\alpha}_s}{2\pi}\int_z \mathcal{M}_{\mathbf{xyz}}\left[1 - \frac{\alpha_s}{4\pi}\log\frac{(\mathbf{x}-\mathbf{z})^2}{(\mathbf{x}-\mathbf{y})^2}\log\frac{(\mathbf{z}-\mathbf{y})^2}{(\mathbf{x}-\mathbf{y})^2}\right](S_{\mathbf{xz}}S_{\mathbf{zy}} - S_{\mathbf{xy}})$$
$$+ \text{other terms}$$

Issue: if $|\mathbf{x}-\mathbf{y}| \ll 1/Q_s$, the [...] term goes negative for $|\mathbf{x}-\mathbf{y}| \ll |\mathbf{x}-\mathbf{z}| \approx |\mathbf{z}-\mathbf{y}| \ll 1/Q_s$.

A problematic term

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$$|\mathbf{x}-\mathbf{y}| \ll |\mathbf{x}-\mathbf{z}| \approx |\mathbf{z}-\mathbf{y}| \ll 1/Q_s.$$

consider the "double-logarithmic approximation" (DLA):
$Y,\ \rho = \log(1/(\mathbf{x}-\mathbf{y})^2 Q_0^2) = \log(1/r^2 Q_0^2)$ both large

Naive Double-logarithmic approximation

$$\partial_Y T_{\mathbf{xy}}(Y) = \frac{\bar{\alpha}_s}{2\pi} \int_{\mathbf{z}} \frac{(\mathbf{x}-\mathbf{y})^2}{(\mathbf{x}-\mathbf{z})^2(\mathbf{z}-\mathbf{y})^2} \left(T_{\mathbf{xz}}(Y) + T_{\mathbf{yz}}(Y) - T_{\mathbf{xy}}(Y) \right)$$

Double-log limit: $(\mathbf{x}-\mathbf{y})^2 Q_0^2 \ll 1$,

- $r \equiv |\mathbf{x}-\mathbf{y}| \ll z \equiv |\mathbf{z}-\mathbf{x}| \approx |\mathbf{z}-\mathbf{y}| \ll 1/Q_0$
- simplify kernel: $\mathcal{M}_{\mathbf{xyz}} \approx r^2/z^4$
- simplify amplitude: $T_{\mathbf{xy}}(Y) = r^2 Q_0^2 \mathcal{A}(z, Y)$
- logarithmic phase-space for contribution from larger dipoles

$$\Rightarrow \qquad \mathcal{A}(r, Y) = \mathcal{A}(r, 0) + \int_0^Y dY_1 \int_{r^2}^{1/Q_0^2} \frac{dz^2}{z^2} \mathcal{A}(z, Y_1)$$

$$= \sum_{n=0}^{\infty} c_n (Y\rho)^n \qquad \text{with } \rho = \log(1/r^2 Q_0^2)$$

Full DLA structure

We get the following structure at DLA (two powers of ρ or Y for each $\bar{\alpha}_s$):

$\bar{\alpha}_s(\rho Y)$			
$\bar{\alpha}_s^2(\rho Y)^2$	$\bar{\alpha}_s^2(\rho Y)(\rho^2)$		
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LL BFKL/BK			

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BFKL/BK	BFKL/BK		

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LL BFKL/BK	NLL BFKL/BK	missing DLA in NLL BFKL/BK	
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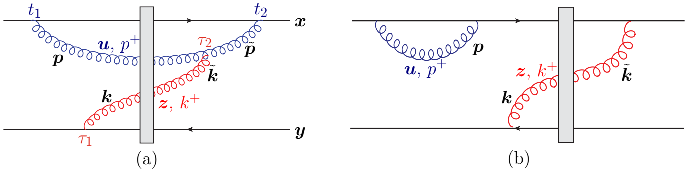
Need for further resummation of transverse logarithms

Following plan

- Known and solved in BFKL since 10-15 years (in Mellin space)
- We look for
  - a solution working with saturation
  - a "diagrammatic" interpretation
- Our next steps
  - DLA $\leftrightarrow$ Time-ordering
  - New DLA evolution equation
  - Improved BK equation



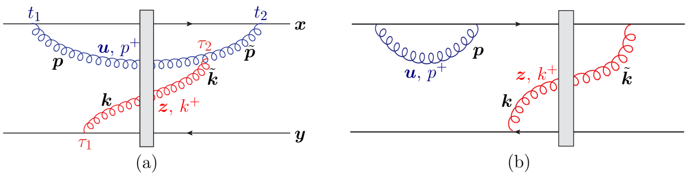
- first emission: gluon p (momentum p^+, coordinate $\mathbf{u}$)
- second emission: gluon k (momentum $k^+ < p^+$, coordinate $\mathbf{z}$)
- $r \equiv |\mathbf{x}-\mathbf{y}| \ll u \equiv |\mathbf{x}-\mathbf{u}| \approx |\mathbf{y}-\mathbf{u}| \ll z \equiv |\mathbf{z}-\mathbf{x}| \approx |\mathbf{z}-\mathbf{y}| \approx |\mathbf{z}-\mathbf{u}|$

(a) contains

$$\left(\frac{\bar{\alpha}_s}{2\pi}\right)^2 \int_{q_0^+}^{q^+} \frac{dk^+}{k^+} \int_{k^+}^{q^+} \frac{dp^+}{p^+} \int_{\mathbf{zu}} \int_{\mathbf{pk}} (\cdots) \frac{k^-}{p^- + k^-}$$

with $p^- = \mathbf{p}^2/p^+$

Refined DLA and time-ordering



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with $\tau_p \equiv 1/p^- = p^+/\mathbf{p}^2$, the **lifetime** of gluon p

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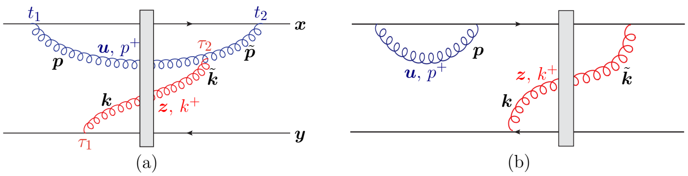

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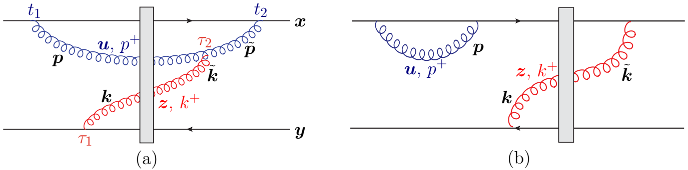

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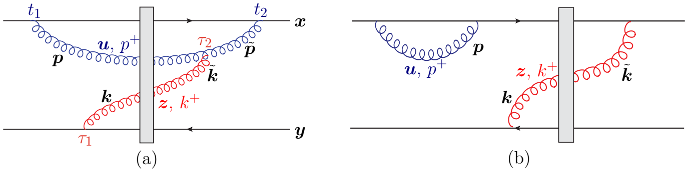

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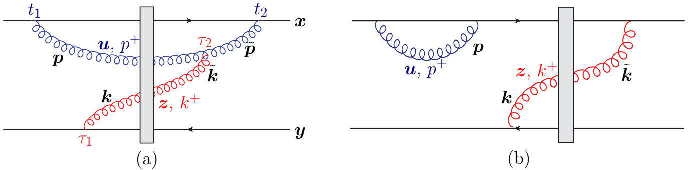
Sum of all "real" **time-ordered p** emissions

$$\left(\frac{\bar{\alpha}_s}{2\pi}\right)^2 \int_{q_0^+}^{q^+} \frac{dk^+}{k^+} \int_{k^+}^{q^+} \frac{dp^+}{p^+} \int_{\mathbf{uz}} \Theta(k^+\mathbf{z}^2 < p^+\mathbf{u}^2)$$
$$(\mathcal{M}_{\mathbf{xyu}}\mathcal{M}_{\mathbf{uyz}}S_{\mathbf{xu}}S_{\mathbf{uz}}S_{\mathbf{zy}} + \mathcal{M}_{\mathbf{xyu}}\mathcal{M}_{\mathbf{xuz}}S_{\mathbf{xz}}S_{\mathbf{uz}}S_{\mathbf{uy}})$$

Refined DLA and time-ordering



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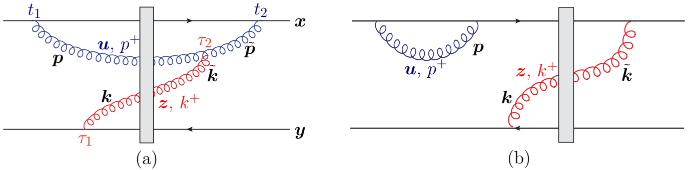
Other real emissions: the sum vanish

Virtual: insert $1 = \Theta(k^+\mathbf{z}^2 < p^+\mathbf{u}^2) + \Theta(k^+\mathbf{z}^2 > p^+\mathbf{u}^2)$

Anti-time-ordered vanish, **time-ordered** give

$$-\left(\frac{\bar{\alpha}_s}{2\pi}\right)^2 \int_{q_0^+}^{q^+} \frac{dk^+}{k^+} \int_{k^+}^{q^+} \frac{dp^+}{p^+} \int_{\mathbf{uz}} \Theta(k^+\mathbf{z}^2 < p^+\mathbf{u}^2) \mathcal{M}_{\mathbf{xyu}} \mathcal{M}_{\mathbf{xyz}} S_{\mathbf{xz}} S_{\mathbf{zy}}$$

Refined DLA and time-ordering



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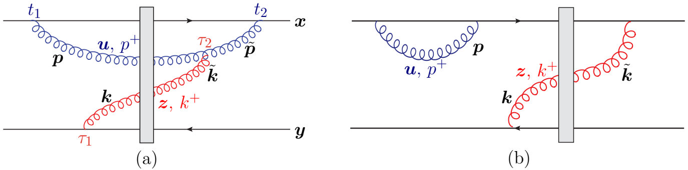
In the end: one needs to impose **time-ordering**

$$\Theta(k^+\mathbf{z} < p^+\mathbf{u})$$

Refined DLA and time-ordering



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Imposing the ordering we get

$$\left(\frac{\bar{\alpha}_s}{2\pi}\right)^2 \int_{q_0^+}^{q^+} \frac{dk^+}{k^+} \int_{k^+}^{q^+} \frac{dp^+}{p^+} \int_{\mathbf{uz}} \Theta(p^+ > k^+(z/u)^2) \frac{r^2}{u^2 z^4} T(z)$$

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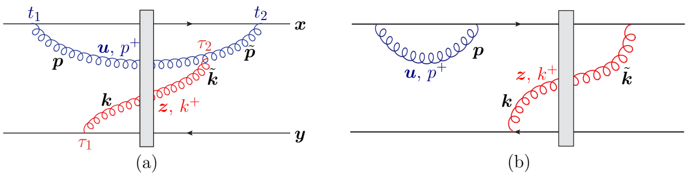

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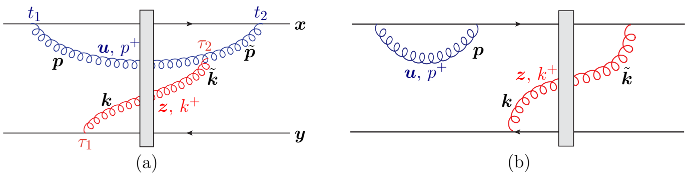
Refined DLA and time-ordering



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Imposing the ordering we get

$$\left(\frac{\bar{\alpha}_s}{2\pi}\right) \int_{q_0^+}^{q^+} \frac{dk^+}{k^+} \int_{\mathbf{z}} \frac{r^2}{z^4} \Big[\underbrace{(\bar{\alpha}_s \rho Y)}_{LL} - \frac{1}{2} \underbrace{(\bar{\alpha}_s \rho^2)}_{NLL} \Big] T(z)$$

coming from the transverse phase-space opening up!

Take-home message

Time-ordering from the "Feynman graphs" ⟷ double-logs

Emission to DLA accuracy

Imposing time ordering gives:

$$\mathcal{A}(r,q^+) = \mathcal{A}(r,0) + \int_{r^2}^{1/Q_0^2} \frac{dz^2}{z^2} \int_{q_0^+}^{q^+(r/z)^2} \frac{dk^+}{k^+} \mathcal{A}(z,k^+)$$

or

$$\mathcal{A}(\rho,Y) = \mathcal{A}(\rho,0) + \int_0^{\rho} d\rho_1 \int_0^{Y+\rho_1-\rho} dY_1\, \mathcal{A}(\rho_1,Y_1)$$
$$= \sum_{n=0}^{\infty} c_n[(Y-\rho)\rho]^n$$

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Interpretation: reduced rapidity phase-space for large transverse gaps

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Interpretation: reduced rapidity phase-space for large transverse gaps

Issue: non-local in rapidity [i.e. not simply $\partial_Y \mathcal{A} = ...$]

Locality in rapidity: solution in Mellin space

Solve for the "Green" function: $\mathcal{A}(\rho, Y) = f(\rho, Y)$ with $f(\rho, 0) = \delta(\rho)$.
For $Y > \rho$, we find

$$f(\rho, Y) = \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \frac{d\xi}{2i\pi} \exp\left[\frac{\bar{\alpha}_s}{1-\xi}(Y-\rho) - (1-\xi)\rho\right]$$

$$= \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \frac{d\gamma}{2i\pi} J(\gamma) \exp\left[\bar{\alpha}_s \chi_{\text{DLA}}(\gamma) Y - (1-\gamma)\rho\right]$$

- Resummed kernel

$$\chi_{\text{DLA}}(\gamma) = \frac{1}{2}\left[-(1-\gamma) + \sqrt{(1-\gamma)^2 + 4\bar{\alpha}_s}\right] \approx \frac{\bar{\alpha}_s}{1-\gamma} - \frac{\bar{\alpha}_s^2}{(1-\gamma)^3} + \dots$$

- Jacobian $J(\gamma) \longrightarrow$ resummed impact factor
- guarantees that f satisfies a *local* equation in Y

Local resummed evolution

Going back to coordinate space, we find, still for $Y > \rho$

$$f(\rho, Y) = \tilde{f}(\rho, 0) + \int_0^Y dY_1 \int_0^\rho d\rho_1 \mathcal{K}_{\text{DLA}}(\rho - \rho_1) f(\rho_1, Y_1),$$

with the **resummed kernel**

$$\boxed{\mathcal{K}_{\text{DLA}}(\rho) = \frac{J_1(2\sqrt{\bar{\alpha}_s \rho^2})}{\sqrt{\bar{\alpha}_s \rho^2}} \approx 1 - \frac{\bar{\alpha}_s \rho^2}{2} + \frac{(\bar{\alpha}_s \rho^2)^2}{12} + \dots}$$

- Initial condition $\tilde{f}$ includes resummation of "J"
- similar equation (with the same kernel) for $\mathcal{A}$.

Extension to BK evolution

Can be straightforwardly generalised to BK:

- Work with the dipole amplitude $T = e^{-\rho}\mathcal{A}$
- use transverse coordinates: $\rho \to \log(1/Q_0^2 r^2)$, $T(Y,\rho) \to T_{\mathbf{xy}}(Y), \dots$
- restore the full kernel: $(r^2/z^4)dz^2 \to \mathcal{M}_{\mathbf{xyz}} d^2z$
- replace the argument of $\mathcal{K}_{\text{DLA}}$:

$$\rho - \rho_1 \to \log(z^2/r^2) \to \sqrt{\log\frac{(\mathbf{x}-\mathbf{z})^2}{(\mathbf{x}-\mathbf{y})^2}\log\frac{(\mathbf{z}-\mathbf{y})^2}{(\mathbf{x}-\mathbf{y})^2}}$$

Improved BK evolution

Collinearly-improved BK evolution:

$$\partial_Y T_{\mathbf{xy}} = \frac{\bar{\alpha}_s}{2\pi} \int d^2\mathbf{z} \mathcal{M}_{\mathbf{xyz}} \mathcal{K}_{\text{DLA}}(L) \left[T_{\mathbf{xz}} + T_{\mathbf{zy}} - T_{\mathbf{xy}} \right],$$

$$\mathcal{K}_{\text{DLA}}(L) = \frac{J_1(2\sqrt{\bar{\alpha}_s L})}{\sqrt{\bar{\alpha}_s L 2}}$$

$$L = \sqrt{\log \frac{(\mathbf{x}-\mathbf{z})^2}{(\mathbf{x}-\mathbf{y})^2} \log \frac{(\mathbf{z}-\mathbf{y})^2}{(\mathbf{x}-\mathbf{y})^2}}$$

Running coupling and single logarithms

[Not discussed in details here]

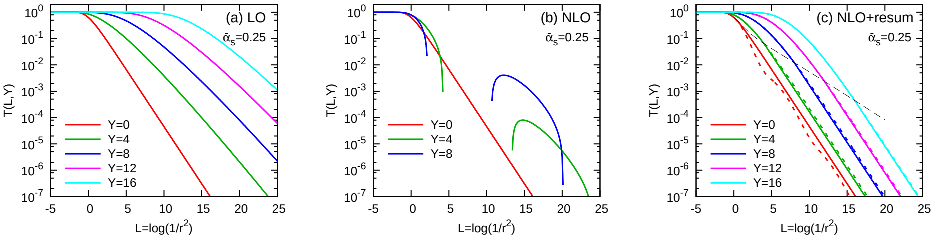
There are other possibly-large corrections:

- Single transverse logarithms
  - can be extracted from the double integration in NLO BK
  - related to “hard” splittings
  - collinear and “anti-”collinear
  - can be (partially) resummed
- running-coupling corrections: different prescriptions
  - Balitsky
  - smallest dipole
  - “fastest apparent convergence”: absorb all the $\bar{\alpha}_s^2 \bar{b}$ terms in NLO BK

Once these have been properly resummed the remaining $\mathcal{O}(\bar{\alpha}_s^2)$ corrections are a genuine small (NLO) correction

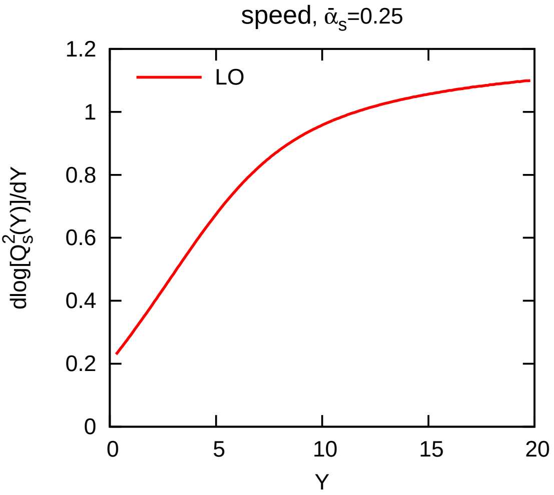
Stabilised evolution

[IMMST,15]



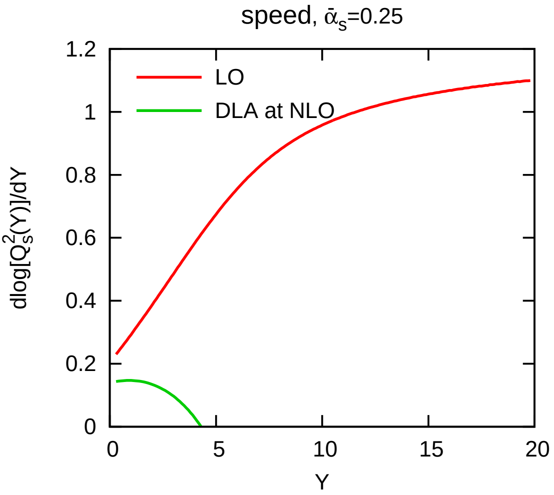
Evolution clearly stabilised

Evolution speed



[IMMST,15]

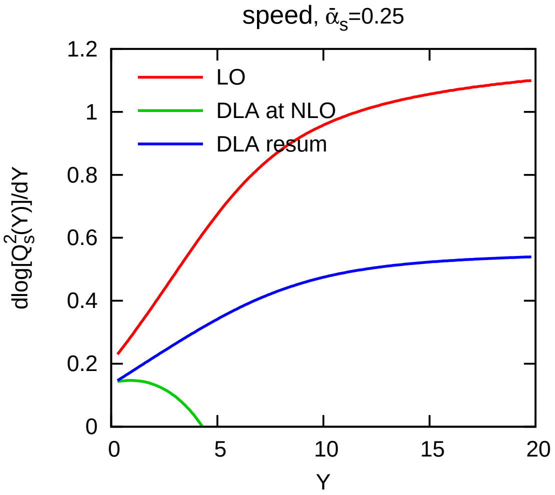
Evolution speed



[IMMST,15]

- unstable at NLO

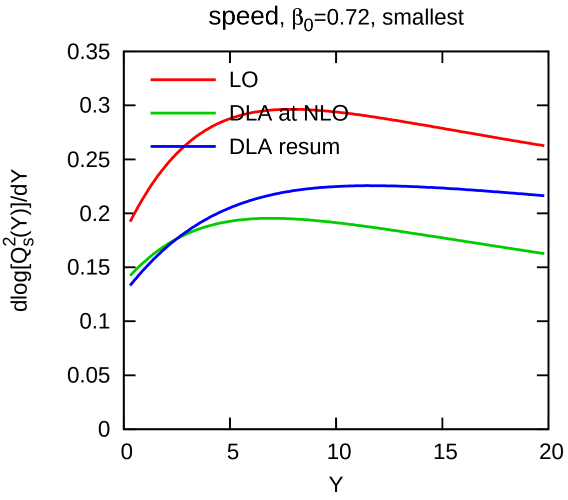
Evolution speed



[IMMST,15]

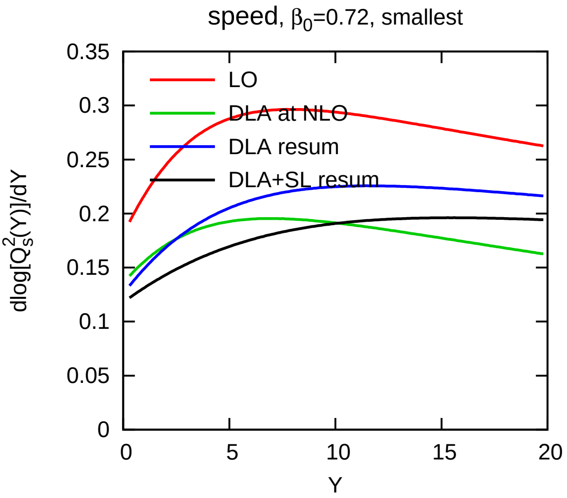
- unstable at NLO
- DLA-resum stabilised

Evolution speed



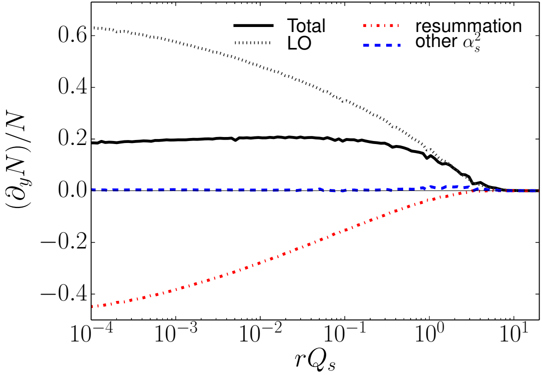
[IMMST,15]

- unstable at NLO
- DLA-resum stabilised
- with running-coupling
- slower than LO



[IMMST,15]

- unstable at NLO
- DLA-resum stabilised
- with running-coupling
- slower than LO
- single-logs also matter



[IMMST,15]

- unstable at NLO
- DLA-resum stabilised
- with running-coupling
- slower than LO
- single-logs also matter
- remaining NLO corrections are small

[Lappi,Mäntysaari,16]

What about describing HERA data?

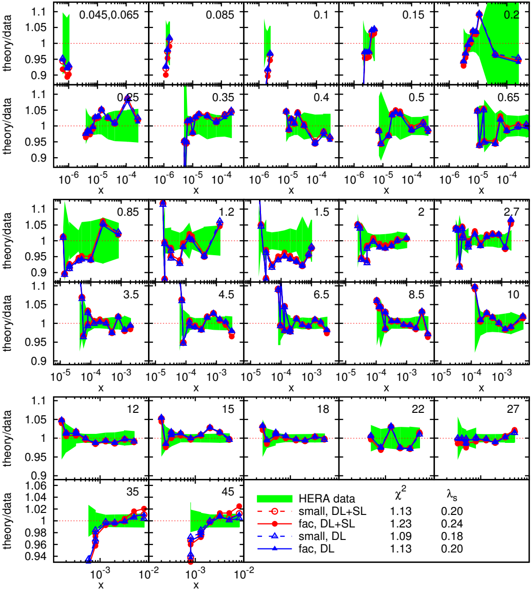
Improved BK vs. data

- Setup: (variations tested)
  - Evolution: LO + resummed double logs + resummed single logs
  - Running coupling: smallest dipole or "fac"
  - Quark masses: $m_{u,d,s} = 100$ MeV, $m_c = 1.3$ GeV
  - Fit $\tilde{\sigma}_r$, check F_L and $\tilde{\sigma}_r^{(c\bar{c})}$
  - $x < 0.01$, $Q^2 < 50$ GeV
  - Various initial conditions including

$$T(Y_0, r) = \left\{1 - \exp\left[-\left(\frac{r^2 Q_0^2}{4}\,\bar{\alpha}_s(r)\left[1 + \ln\left(\frac{\bar{\alpha}_{\rm sat}}{\bar{\alpha}_s(r)}\right)\right]\right)^p\right]\right\}^{1/p}$$

- Main observations:
  - Overall good χ^2 ($\sim 1.1 - 1.2$)
  - Parameters take acceptable values

Improved BK vs. data



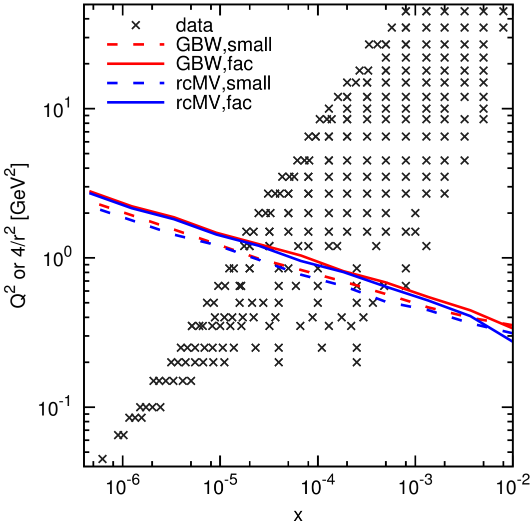
	χ^2	λ_s
HERA data		
small, DL+SL	1.13	0.20
fac, DL+SL	1.23	0.24
small, DL	1.09	0.18
fac, DL	1.13	0.20

Conclusions and perspectives

- Take-home message(s):
  - "raw" (LO,NLO,...) BK unstable due to large transverse logs
  - Resumming these logs ⇒ improved BK
    First predictive version of BK
  - DLA resummation ↔ time ordering
  - Nice description of HERA data
- Open questions:
  1. Include finite NLO corrections
  2. Fourier transform ↔ particle production?
  3. NLO+resummation for impact factors?
  4. Uncertainties?

Backup

Saturation scale



Fit parameters

init cdt.	RC schm	sing. logs	χ^2 per data point			parameters			
			$\sigma_{\rm red}$	$\sigma_{\rm red}^{cc}$	F_L	R_p[fm]	Q_0[GeV]	C_α	p
GBW	small	yes	1.135	0.552	0.596	0.699	0.428	2.358	2.802
GBW	fac	yes	1.262	0.626	0.602	0.671	0.460	0.479	1.148
rcMV	small	yes	1.126	0.565	0.592	0.707	0.633	2.586	0.807
rcMV	fac	yes	1.228	0.647	0.594	0.677	0.621	0.504	0.541
GBW	small	no	1.121	0.597	0.597	0.716	0.414	6.428	4.000
GBW	fac	no	1.164	0.609	0.594	0.697	0.429	1.195	4.000
rcMV	small	no	1.093	0.539	0.594	0.718	0.647	7.012	1.061
rcMV	fac	no	1.132	0.550	0.591	0.699	0.604	1.295	0.820

High Q^2

init cdt.	RC schm	sing. logs	χ^2/npts for $Q^2_{\rm max}$ 50	100	200	400
GBW	small	yes	1.135	1.172	1.355	1.537
GBW	fac	yes	1.262	1.360	1.654	1.899
rcMV	small	yes	1.126	1.170	1.182	1.197
rcMV	fac	yes	1.228	1.304	1.377	1.421
GBW	small	no	1.121	1.131	1.317	1.487
GBW	fac	no	1.164	1.203	1.421	1.622
rcMV	small	no	1.093	1.116	1.106	1.109
rcMV	fac	no	1.131	1.181	1.171	1.171