QCD at high-energy

Saturation and fluctuation effects

Grégory Soyez

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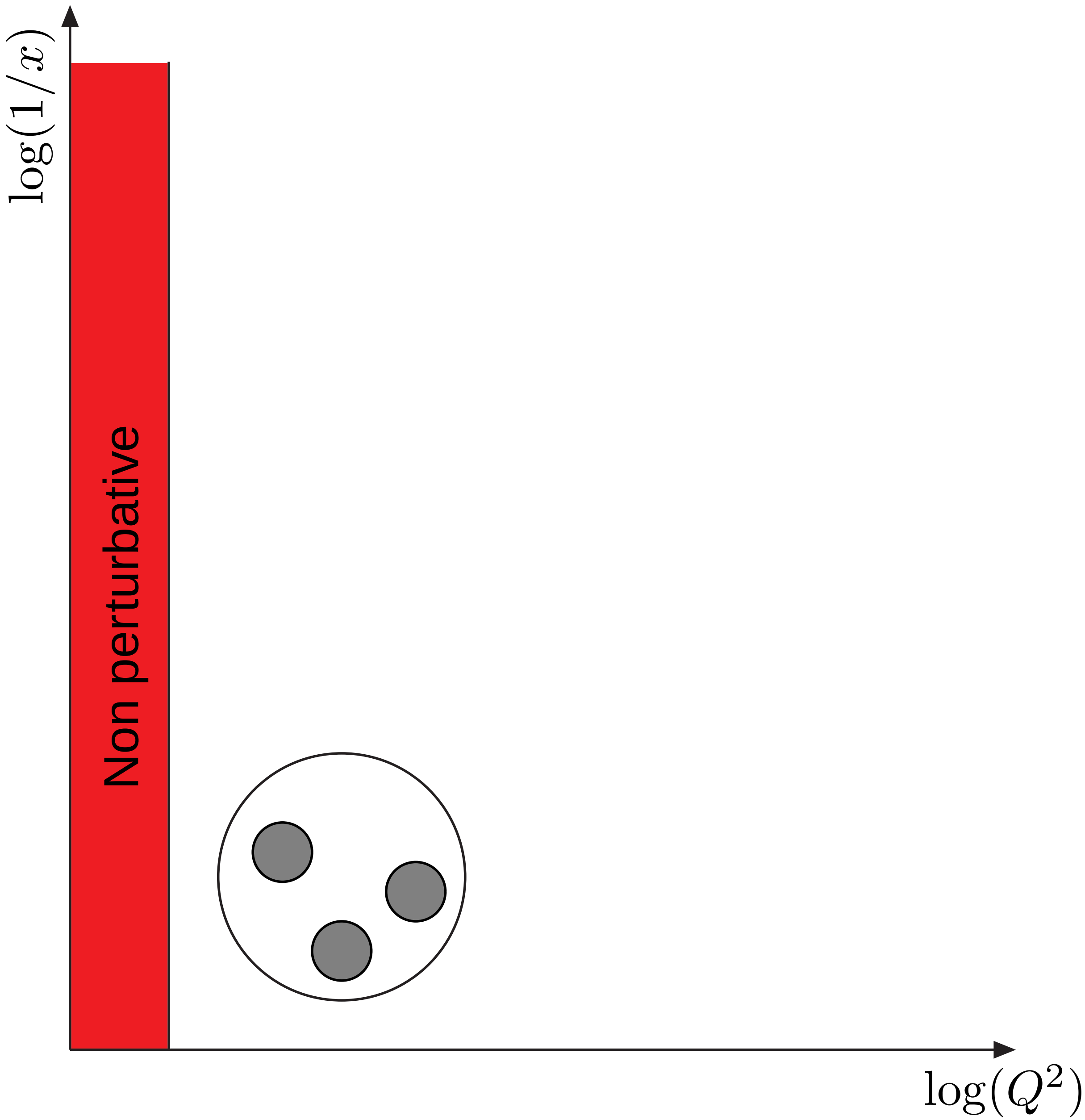
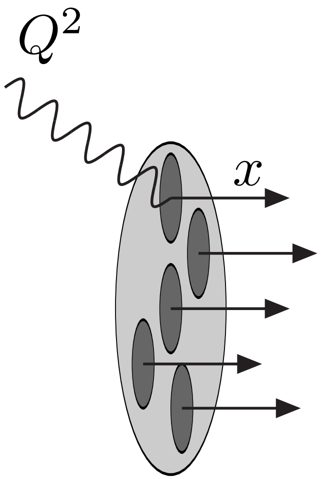
Outline

- Lecture 1: Evolution towards high-energy
  - Motivation
  - Leading log approximation: BFKL equation
  - Saturation effects: BK equation
  - fluctuations
- Lecture 2: Properties of the scattering amplitudes
  - BK, statistical physics and geometric scaling
  - fluctuations, reaction-diffusion and diffusive scaling
  - Applications

Evolution towards high-energy

Motivation

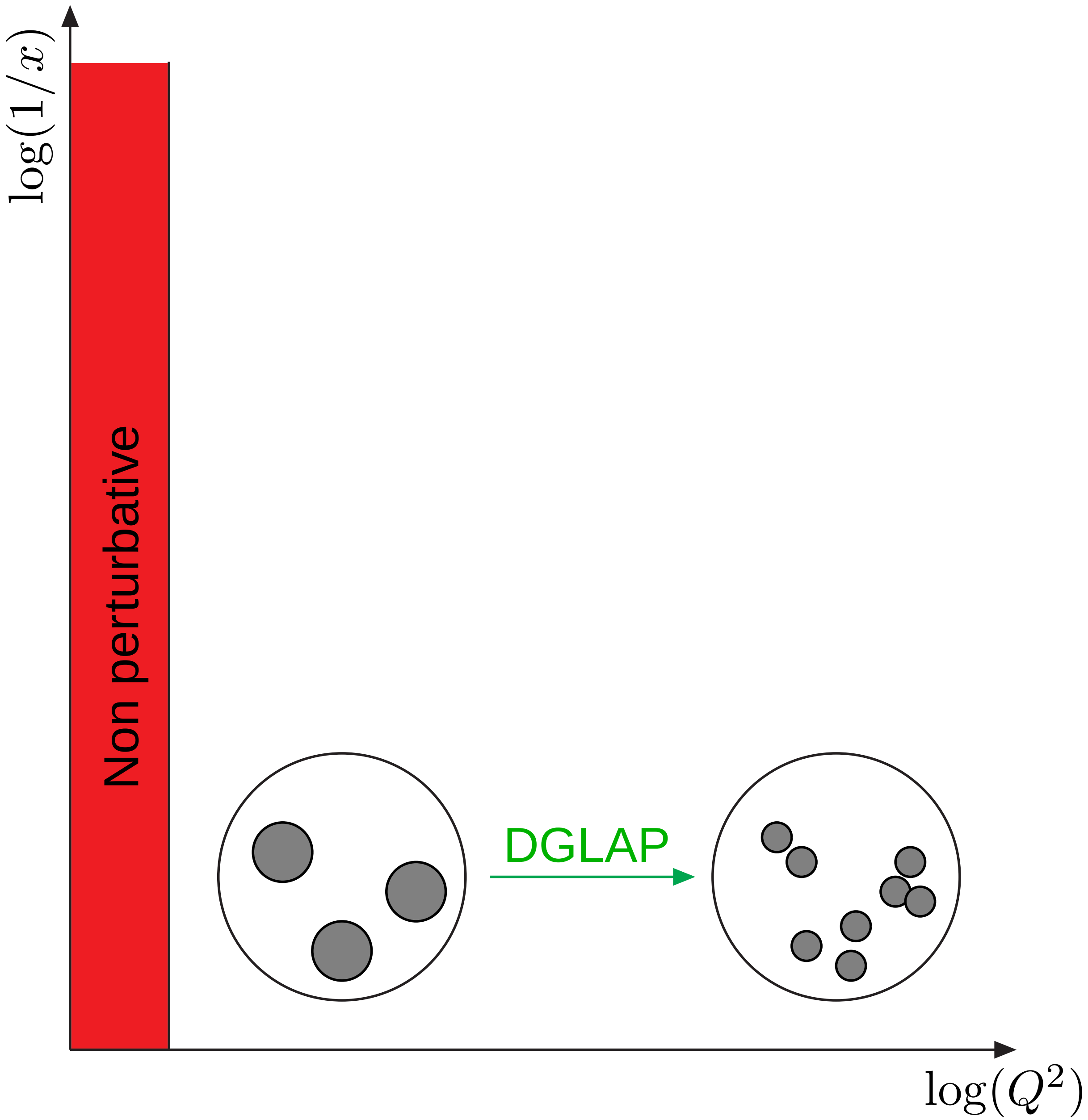


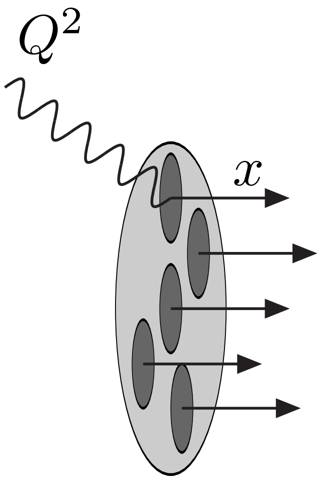


Energy $W^2 = Q^2/x$

Rapidity $Y = \log(1/x)$

Motivation





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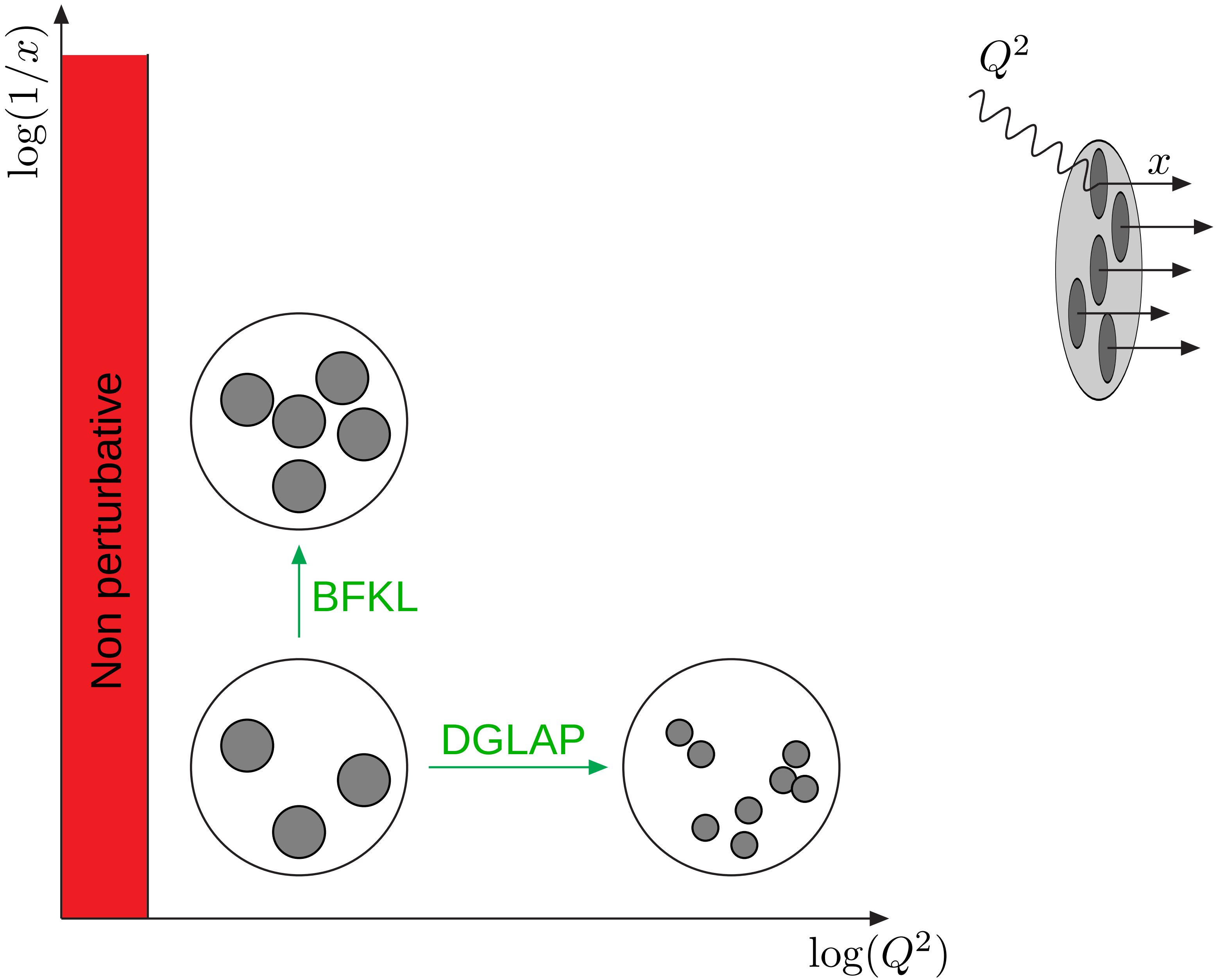

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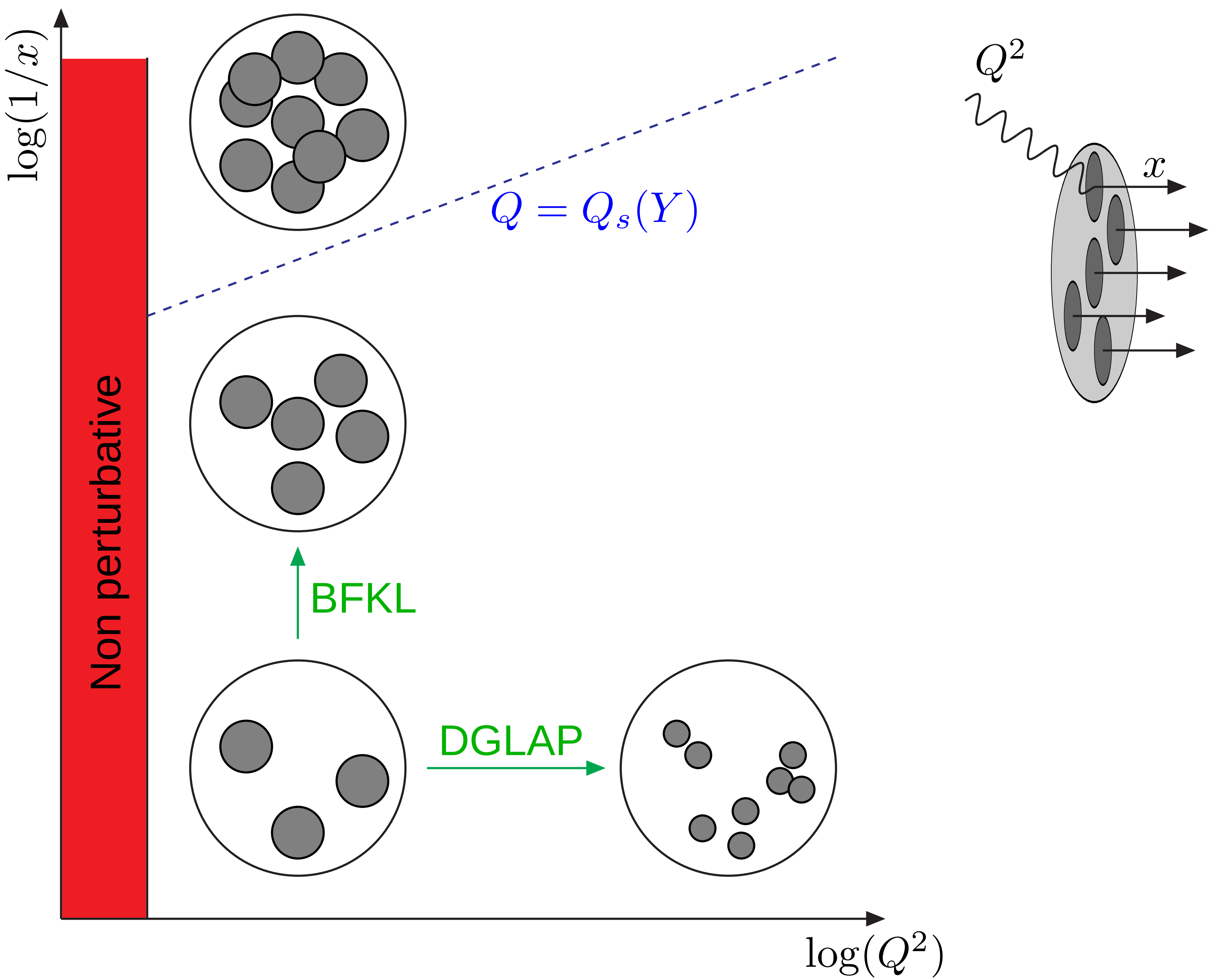

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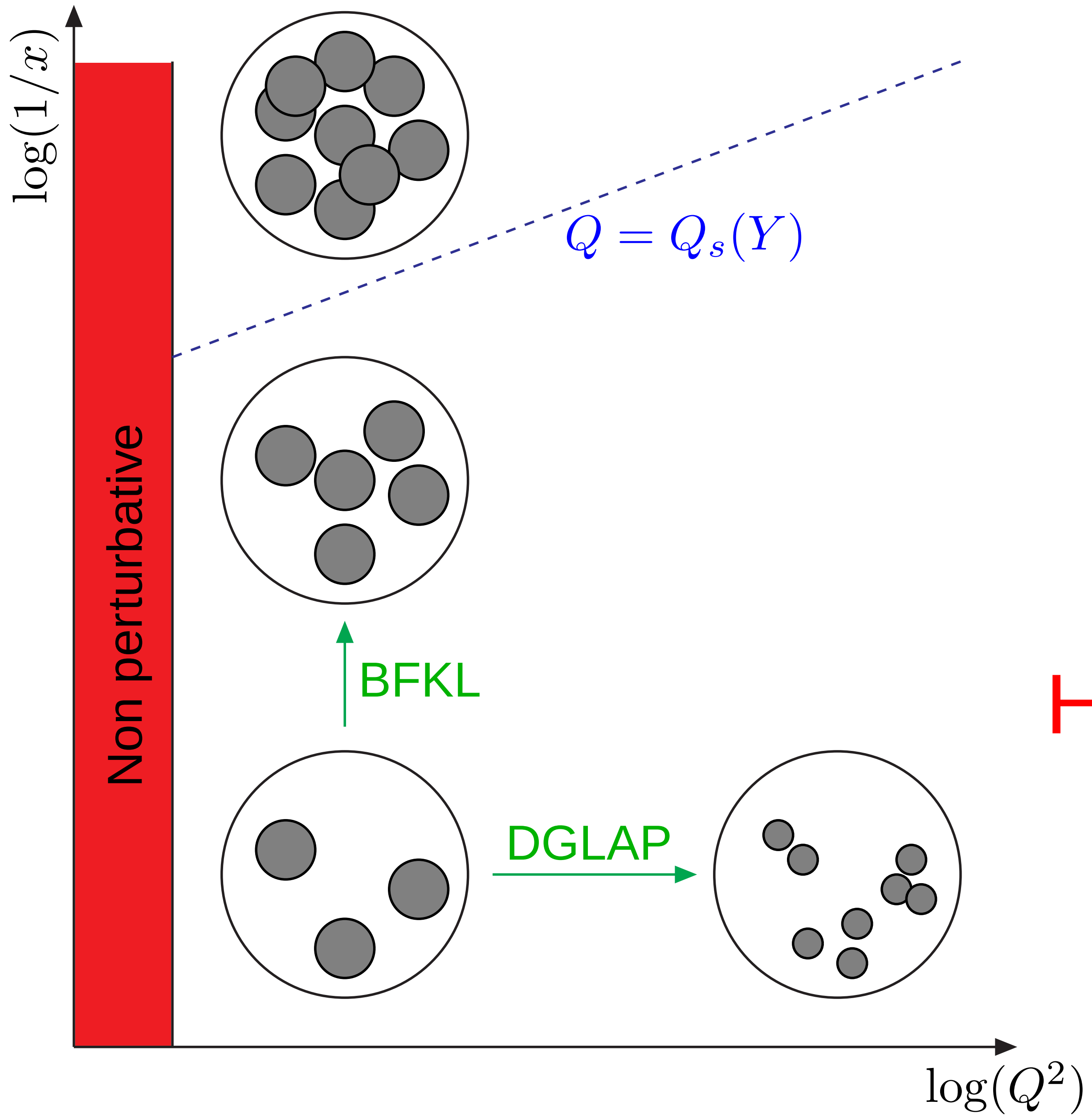
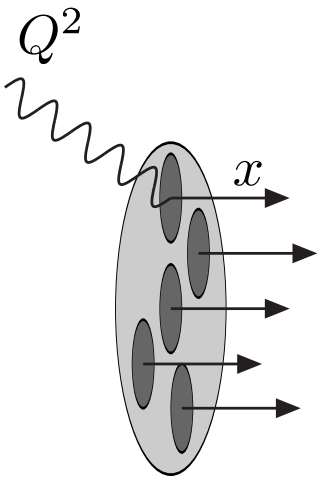




How to describe this in QCD ?

Rough estimates:

Saturation scale: $\leftrightarrow$ partons are covering the proton

$$\underbrace{xg(x,Q^2)}_{\#\text{ gluons}}\ \underbrace{Q^{-2}}_{\text{parton size}} = \underbrace{\pi R_p^2}_{\text{proton size}}$$

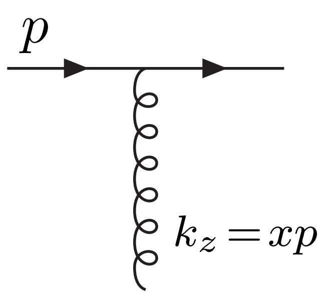
The gluon distribution behaves like: $xg(x,Q^2) \propto x^{-\lambda}$

$$\Rightarrow Q_s^2(Y) \propto R_p^{-2} x^{-\lambda}$$

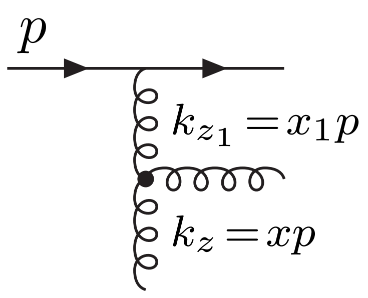
$R_p \approx 1$ fm ≈ 5 GeV^{-1}

$$Q_s^2 \approx 1\text{GeV}^2 \quad \text{at } x = 10^{-4}$$

Need for some careful treatment because of Bremsstrahlung:



$x \ll 1$



$x \ll x_1 \ll 1$

Probability of emission

$$dP \sim \alpha_s \frac{dk^2}{k^2}\frac{dx}{x}$$

In the small-x limit

$$\int_x^1 \frac{dx_1}{x_1} \sim \alpha_s \log(1/x)$$

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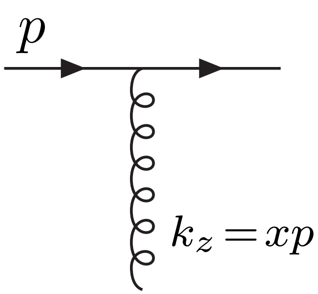

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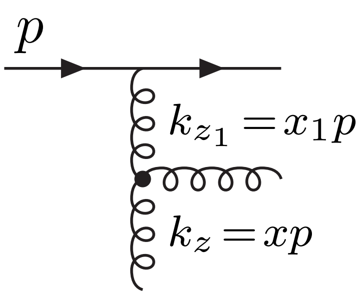

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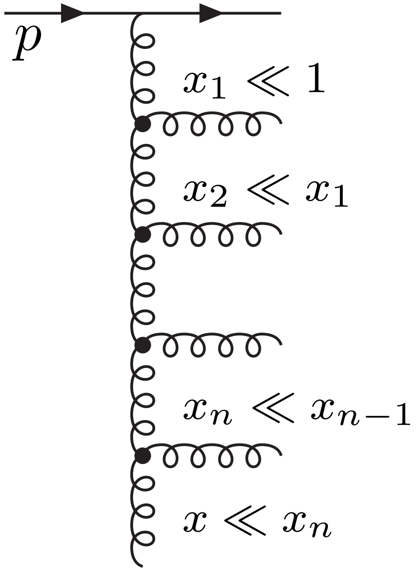


n-gluon emission:

$$\int_x^1 \frac{dx_n}{x_n} \int_{x_n}^1 \frac{dx_{n-1}}{x_{n-1}} \dots \int_{x_2}^1 \frac{dx_1}{x_1} \sim \frac{1}{n!} \alpha_s^n \log^n(1/x)$$

$\longrightarrow$ At small x: need to be resummed:

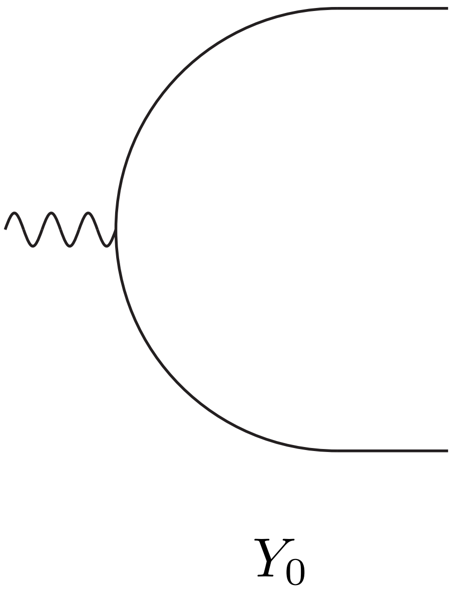
$$\sum_{n=0}^{\infty} \frac{1}{n!} \alpha_s^n \log^n(1/x) \approx e^{\omega Y}$$

Dipole picture

[Mueller, 93]

Consider a fast-moving $q\bar{q}$ dipole

Rapidity: $Y = \log(s)$



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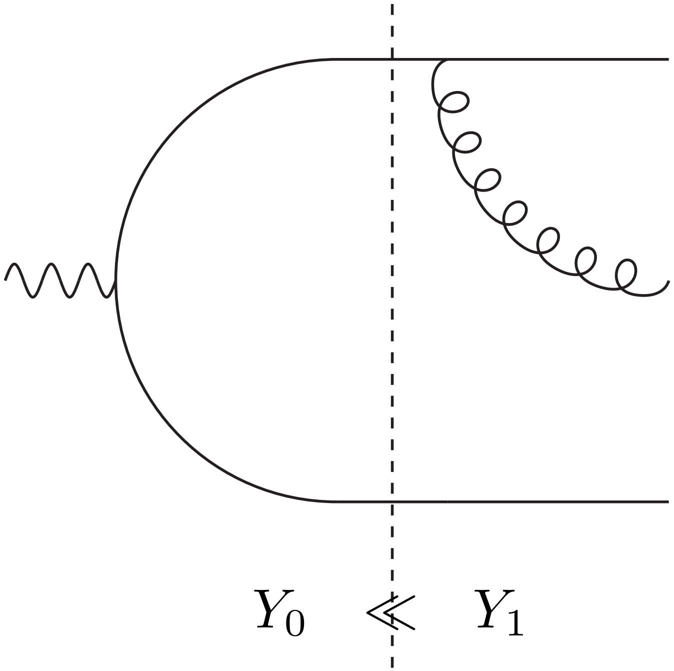

- Probability $\bar{\alpha}K$ of emission

Dipole picture

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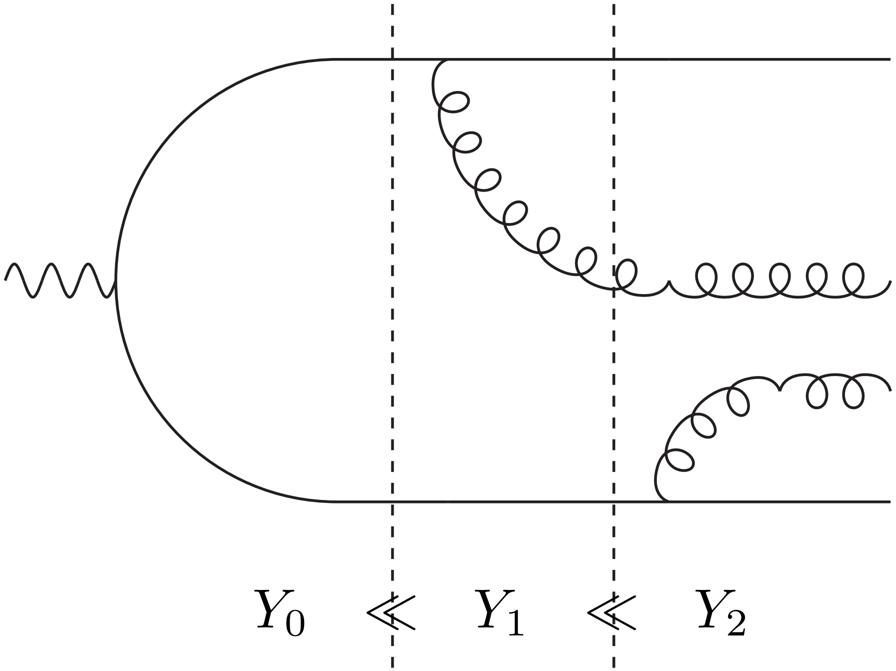

- Probability $\bar{\alpha}K$ of emission
- Independent emissions

Dipole picture

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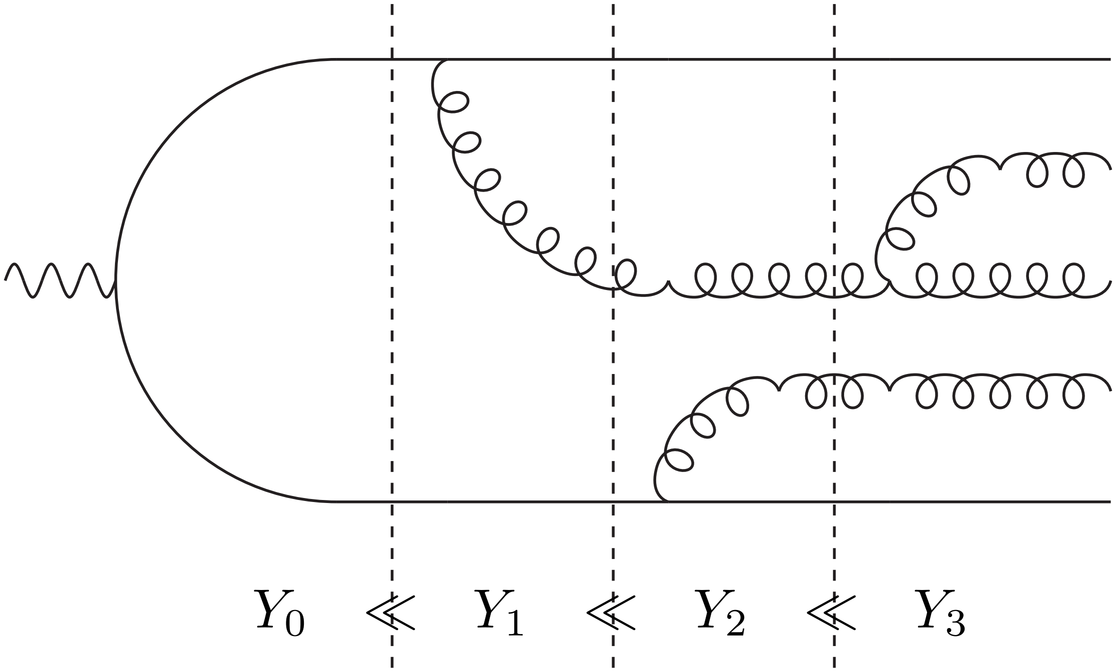

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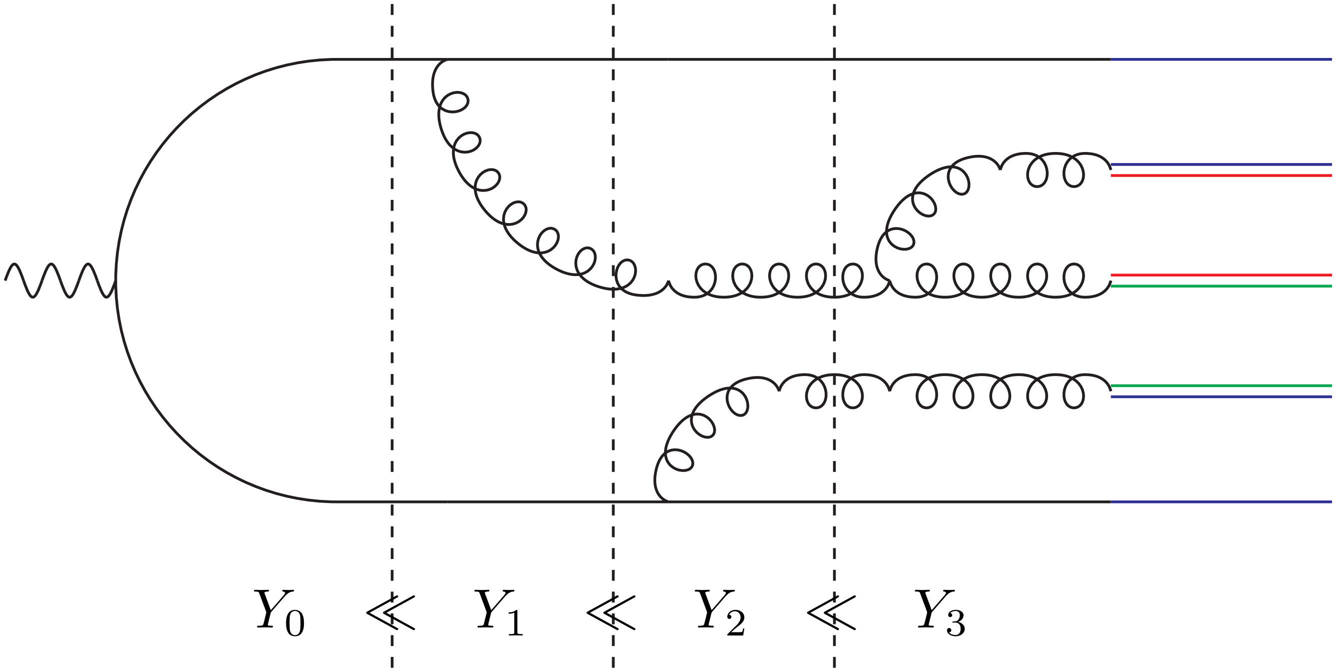
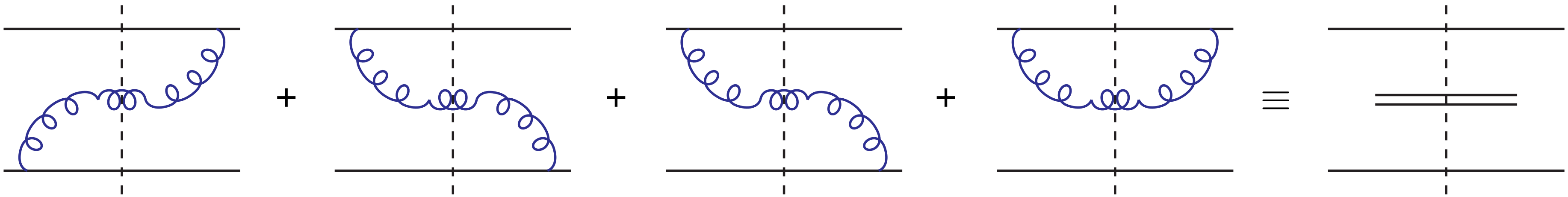


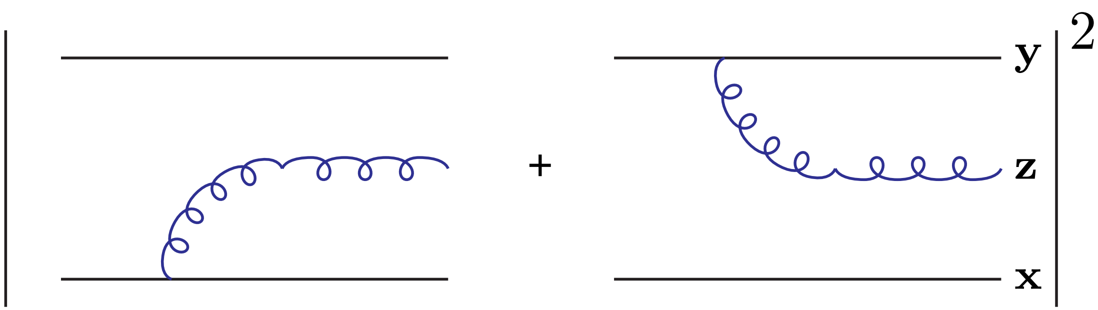
$n(r, Y)$ dipoles

of size r

- Probability $\bar{\alpha}K$ of emission
- Independent emissions
- Large-N_c approximation

Dipole splitting

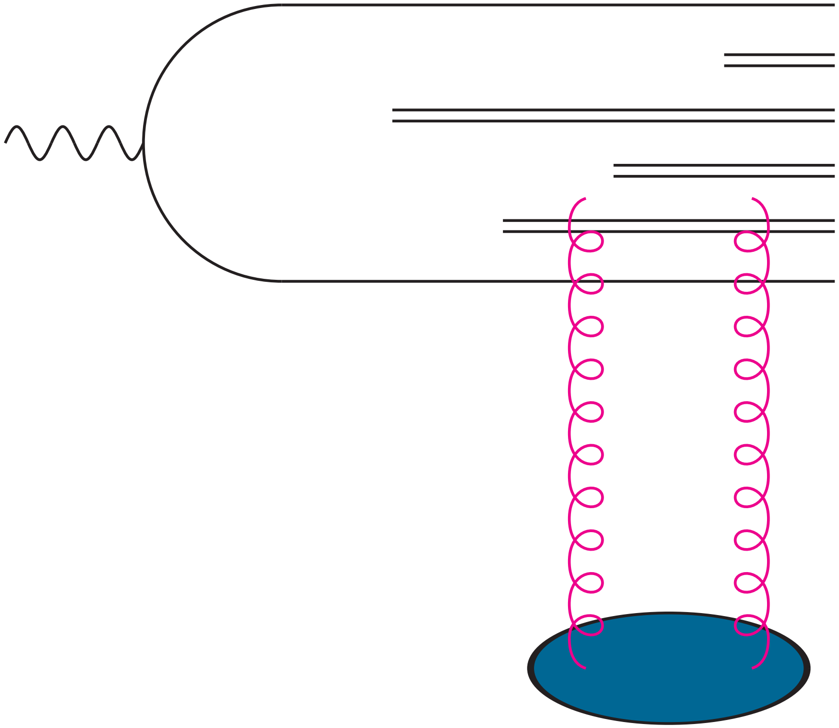




$$= \left[\frac{\mathbf{x}-\mathbf{z}}{(\mathbf{x}-\mathbf{z})^2} - \frac{\mathbf{y}-\mathbf{z}}{(\mathbf{y}-\mathbf{z})^2} \right]^2$$

$$= \frac{\bar{\alpha}}{2\pi} \frac{(\mathbf{x}-\mathbf{y})^2}{(\mathbf{x}-\mathbf{z})^2(\mathbf{z}-\mathbf{y})^2}$$

How to observe this system ?

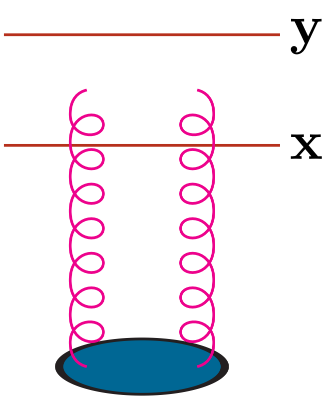


Scattering amplitude

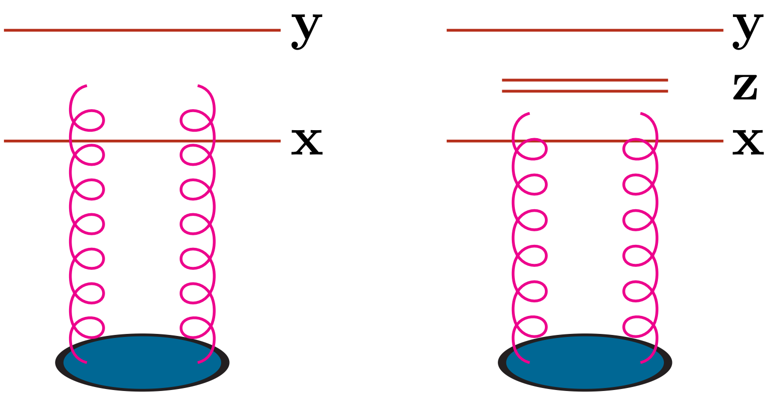
$$T(r, Y) \approx \alpha_s^2 n(r, Y)$$

Count the number of dipoles of a given size

Consider a small increase in rapidity



Consider a small increase in rapidity $\Rightarrow$ splitting

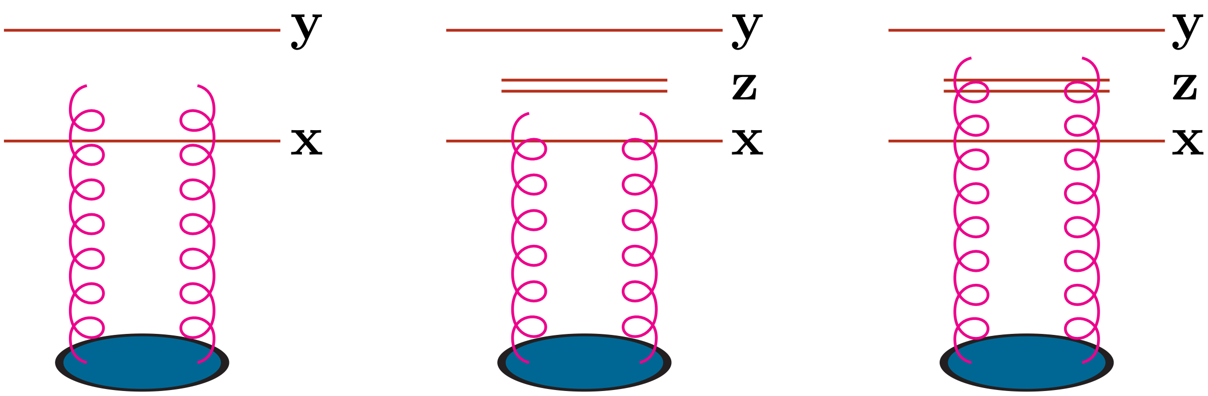


$$\partial_Y T(\mathbf{x}, \mathbf{y}; Y)$$

$$T(\mathbf{x}, \mathbf{z}; Y)$$

BFKL evolution (2/4)

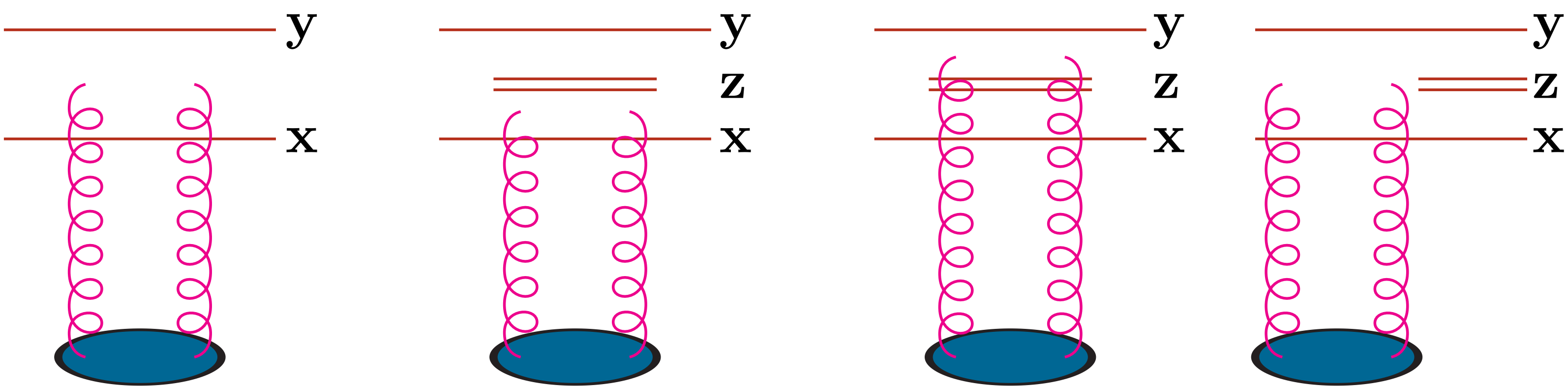
Consider a small increase in rapidity $\Rightarrow$ splitting



$$\partial_Y T(\mathbf{x}, \mathbf{y}; Y)$$

$$T(\mathbf{x}, \mathbf{z}; Y) + T(\mathbf{z}, \mathbf{y}; Y)$$

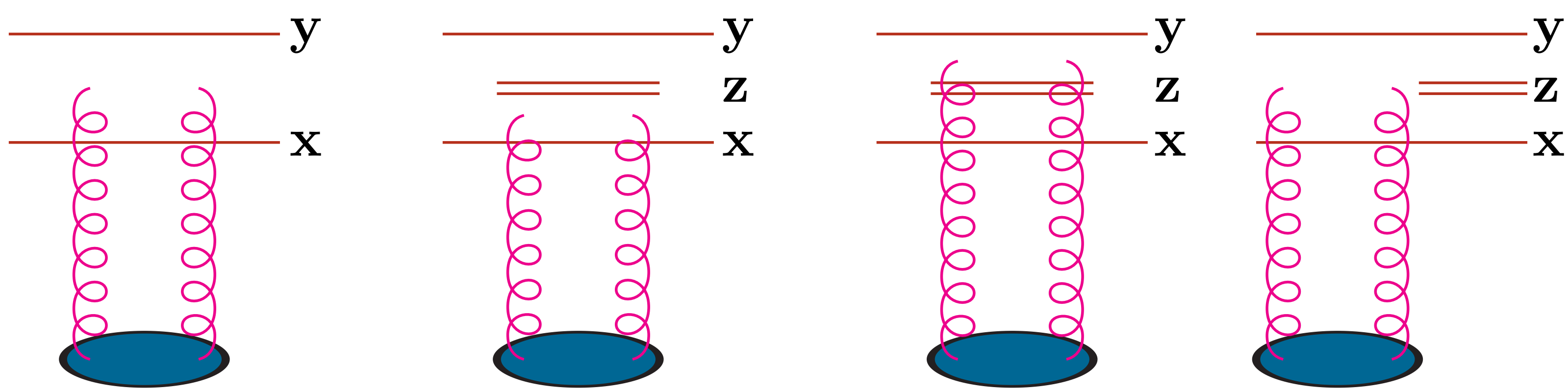
Consider a small increase in rapidity $\Rightarrow$ splitting



$$\partial_Y T(\mathbf{x},\mathbf{y};Y)$$

$$T(\mathbf{x},\mathbf{z};Y) + T(\mathbf{z},\mathbf{y};Y) - T(\mathbf{x},\mathbf{y};Y)$$

Consider a small increase in rapidity $\Rightarrow$ splitting



$$\partial_Y T(\mathbf{x},\mathbf{y};Y) = \bar{\alpha}\int d^2z\,\frac{(\mathbf{x}-\mathbf{y})^2}{(\mathbf{x}-\mathbf{z})^2(\mathbf{z}-\mathbf{y})^2}\left[T(\mathbf{x},\mathbf{z};Y)+T(\mathbf{z},\mathbf{y};Y)-T(\mathbf{x},\mathbf{y};Y)\right]$$

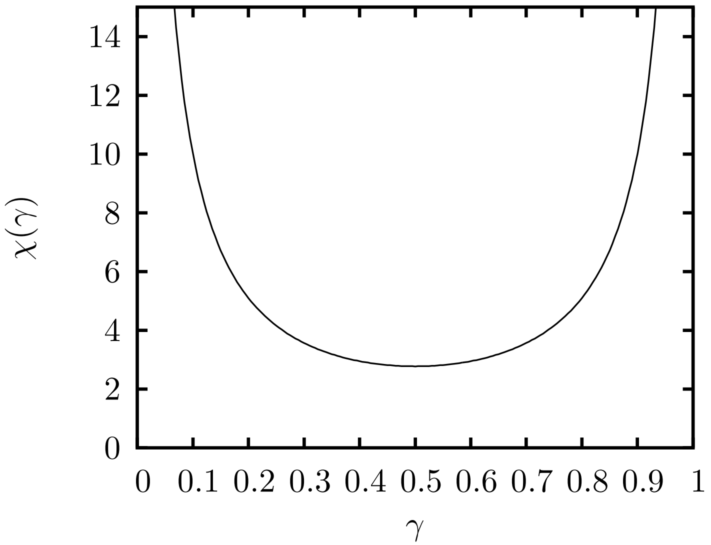
[Balitsky, Fadin, Kuraev, Lipatov, 78]

Solution ? Use Mellin space

$$T(\mathbf{x},\mathbf{y}) = T(|\mathbf{x}-\mathbf{y}|) = T(r) = r^{2\gamma} \qquad \Rightarrow \qquad \partial_Y T(r) = \bar{\alpha}\chi(\gamma)T(r)$$

$\chi(\gamma)$ is the BFKL eigenvalues:

$$\chi(\gamma) = 2\psi(1) - \psi(\gamma) - \psi(1-\gamma)$$



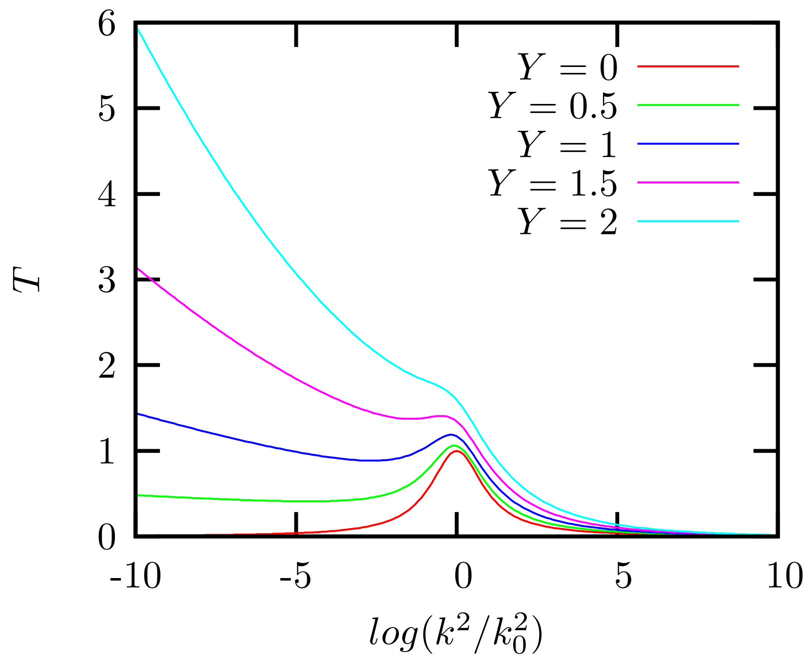
b-independent solution:

$$T(r) = \int_{c-i\infty}^{c+i\infty} \frac{d\gamma}{2i\pi}\, T_0(\gamma)\, \exp\left[\bar{\alpha}\chi(\gamma)Y - \gamma \log\left(\frac{r_0^2}{r^2}\right)\right]$$

Solution in the saddle point approximation

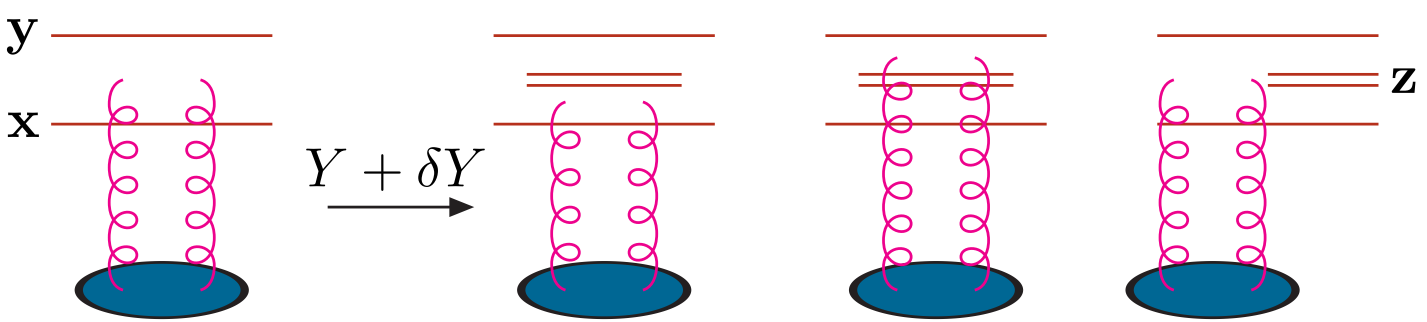
$$T(r,Y) \approx \frac{1}{\sqrt{Y}}\frac{r}{r_0}e^{\omega Y}\ \exp\left[-\frac{\log^2(r^2/r_0^2)}{2\bar{\alpha}\chi''(1/2)Y}\right] \qquad \text{with} \quad \omega = 4\bar{\alpha}\log(2) \approx 0.5$$

- Fast growth of the amplitude
- Intercept value too large
- Same with $r_0/r \to k/k_0$
- problem of diffusion in the infrared
- Violation of the froissart bound:
  $T(Y) \le C \log^2(s)$ $\qquad$ $T(r,b) \le 1$



Let us reconsider one step of the evolution

Rapidity increase $\Rightarrow$ Splitting into 2 dipoles



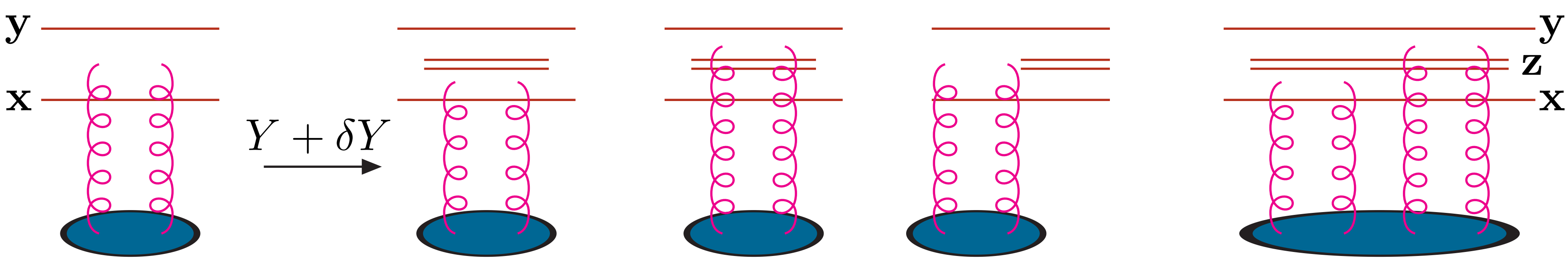
$$\partial_Y \langle T_{\mathbf{xy}} \rangle = \bar{\alpha} \int_z \mathcal{M}_{\mathbf{xyz}} \underbrace{[\langle T_{\mathbf{xz}} \rangle + \langle T_{\mathbf{zy}} \rangle - \langle T_{\mathbf{xy}} \rangle]}_{\text{Linear BFKL}}$$

[Balitsky, Fadin, Kuraev, Lipatov, 78]

Solution: $e^{\omega Y}$ but violates unitarity

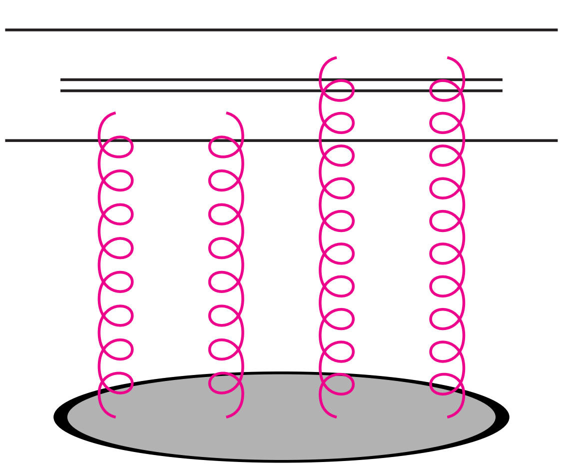
Let us reconsider one step of the evolution

Rapidity increase $\Rightarrow$ Splitting into 2 dipoles



$$\partial_Y \langle T_{\mathbf{xy}} \rangle = \bar{\alpha} \int_z \mathcal{M}_{\mathbf{xyz}} [\underbrace{\langle T_{\mathbf{xz}} \rangle + \langle T_{\mathbf{zy}} \rangle - \langle T_{\mathbf{xy}} \rangle}_{\text{Linear BFKL}} - \underbrace{\langle T_{\mathbf{xz}} T_{\mathbf{zy}} \rangle}_{\text{Unitarity}}]$$

[Balitsky 96]



Proportional to T^2

important when $T \approx 1$

- $\langle T \rangle$, $\langle T^2 \rangle$, ...: JIMWLK/Balitsky equations (at large N_c)
- Mean-field approximation: $\langle T^2 \rangle = \langle T \rangle^2$ (BK equation)

$$\partial_Y \langle T_{\mathbf{xy}} \rangle = \bar{\alpha} \int_z \mathcal{M}_{\mathbf{xyz}} [\underbrace{\langle T_{\mathbf{xz}} \rangle + \langle T_{\mathbf{zy}} \rangle - \langle T_{\mathbf{xy}} \rangle}_{\text{Linear BFKL}} - \underbrace{\langle T_{\mathbf{xz}} \rangle \langle T_{\mathbf{zy}} \rangle}_{\text{Unitarity}}]$$

[Kovchegov 99]

Most simple perturbative evolution equation including BFKL + saturation

Saturation effects: pros

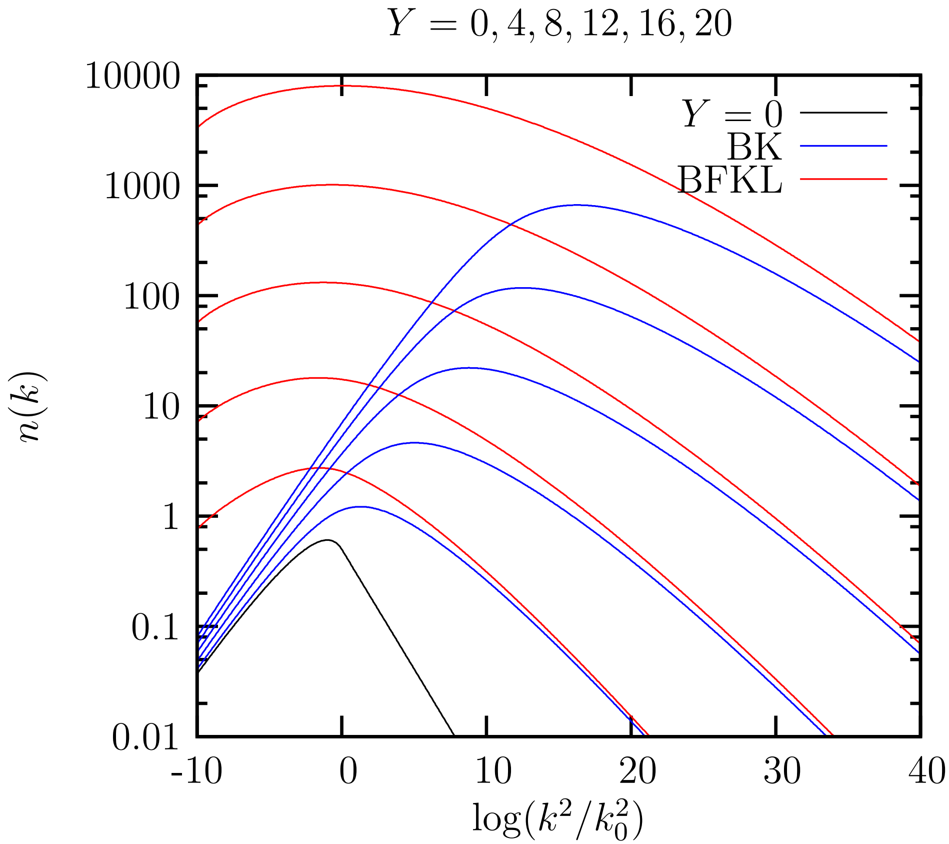
Improvements due to this new term:

- $0 \leq T(\mathbf{x}, \mathbf{y}) \leq 1 \Rightarrow$ unitarity preserved

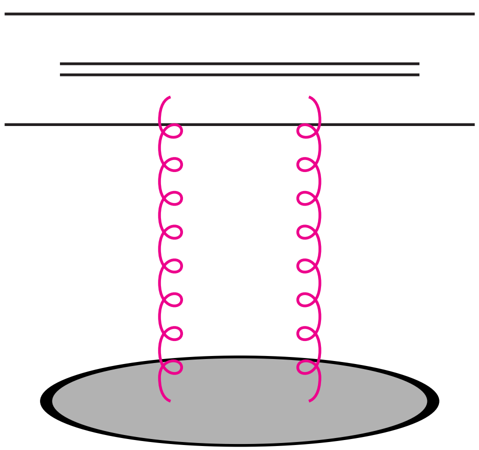
Improvements due to this new term:

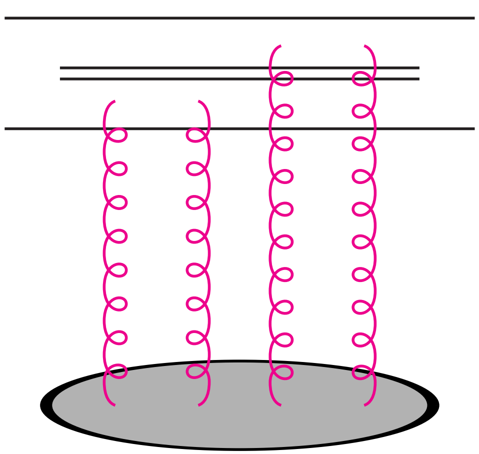
- $0 \leq T(\mathbf{x}, \mathbf{y}) \leq 1 \Rightarrow$ unitarity preserved
- Cut the diffusion to the infrared

unintegrated
gluon distribution



Equivalence with “usual” Feynman graphs:





Saturation effects: fan diagrams

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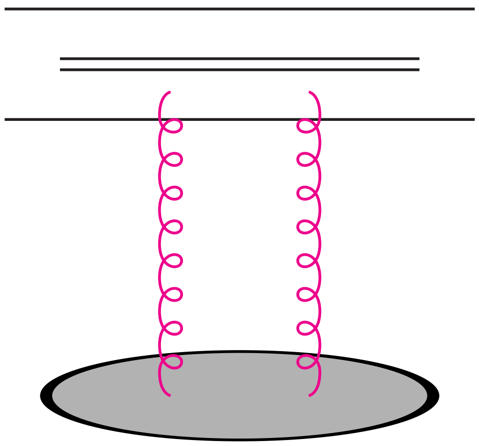

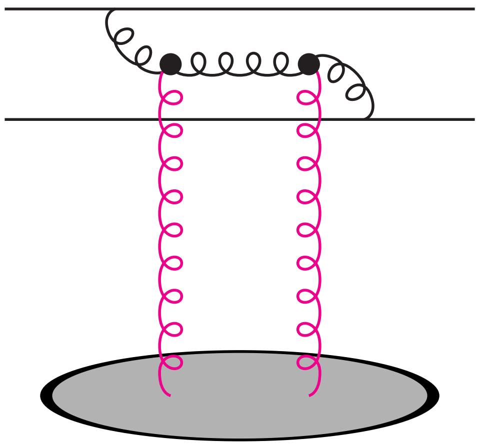

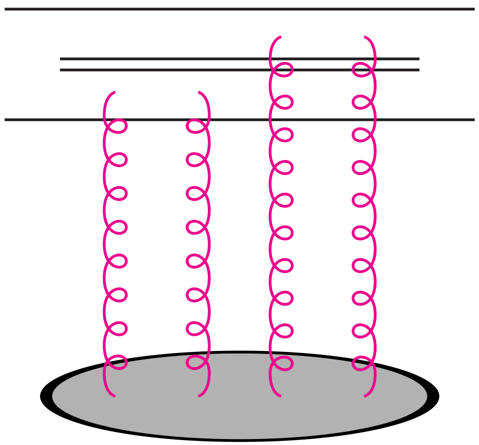

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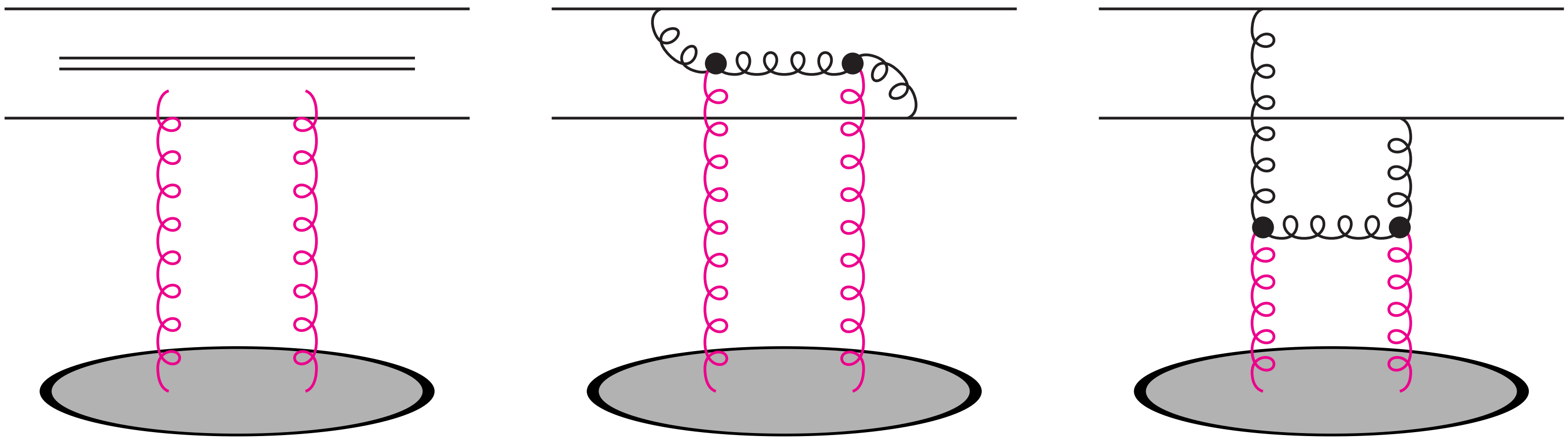
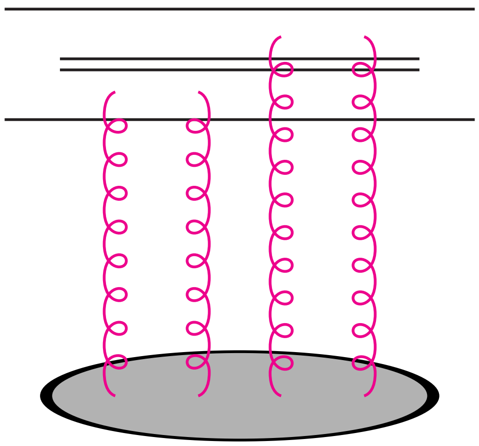


BFKL ladder



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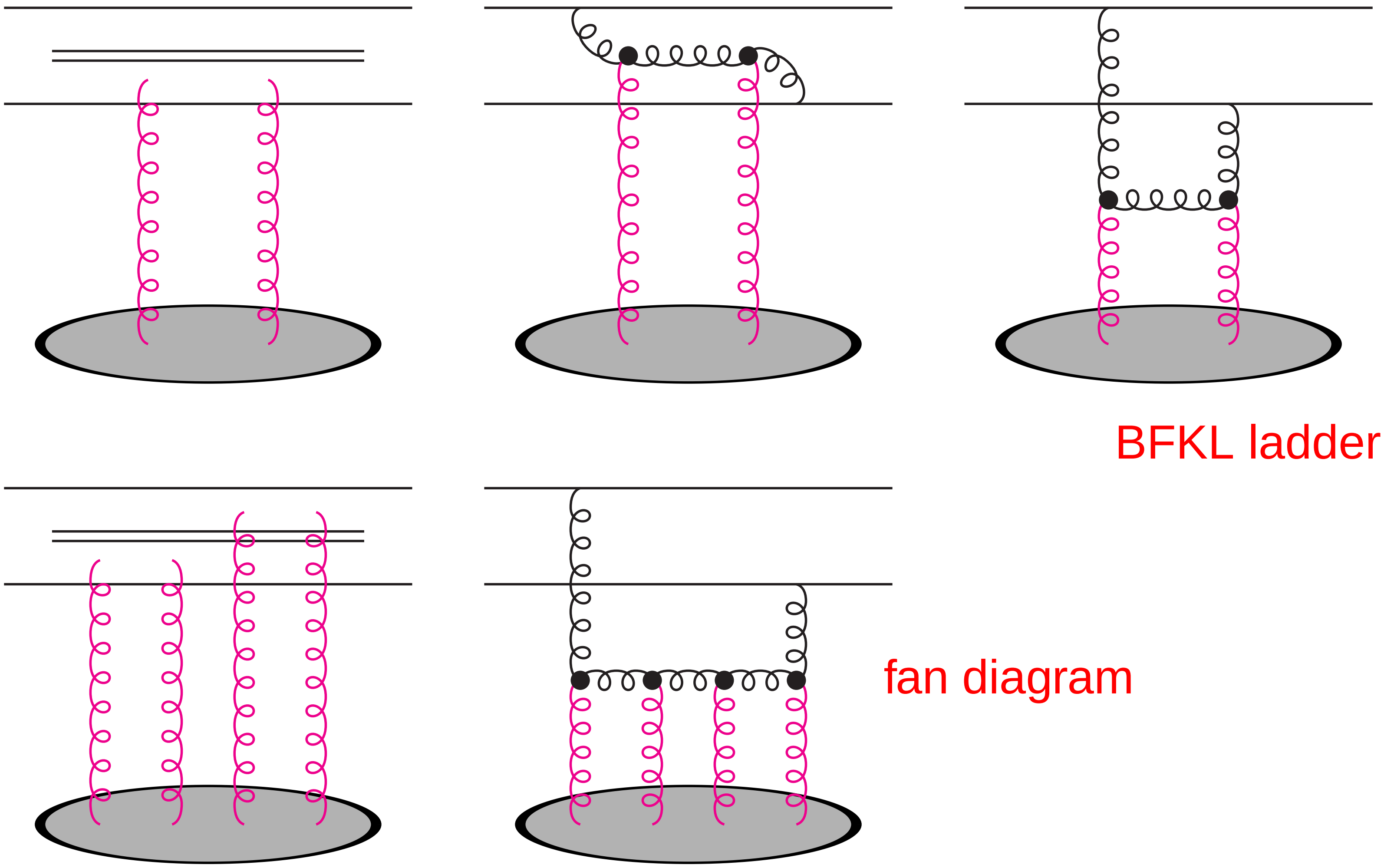
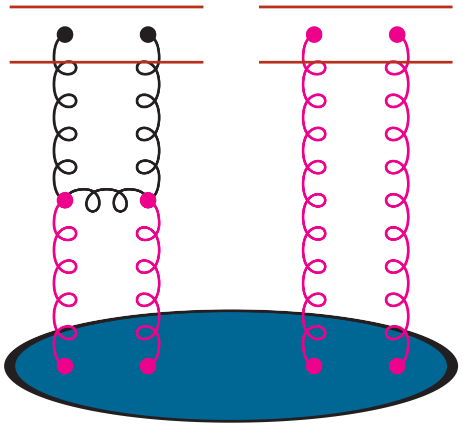


Consider evolution of $\langle T^{(2)} \rangle$

[E. Iancu, D. Triantafyllopoulos, 05]
Also A. Mueller, S. Munier, A. Shoshi, S. Wong



- Usual BFKL ladder

$$\partial_Y \langle T^{(2)} \rangle \propto \langle T^{(2)} \rangle$$

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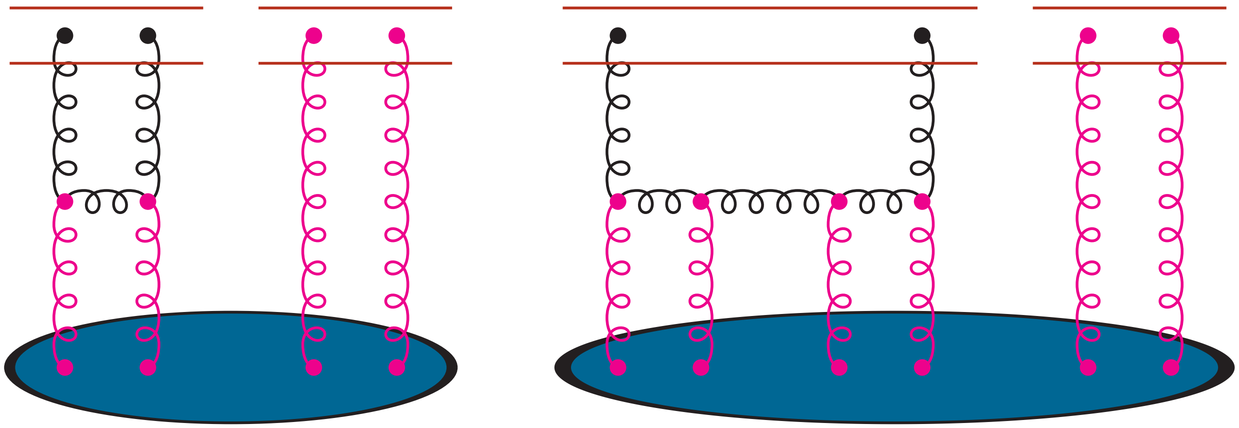

- Usual BFKL ladder $\qquad \partial_Y \left\langle T^{(2)} \right\rangle \propto \left\langle T^{(2)} \right\rangle$
- fan diagram $\longrightarrow$ saturation effects $\qquad \partial_Y \left\langle T^{(2)} \right\rangle \propto \left\langle T^{(3)} \right\rangle$

Consider evolution of $\langle T^{(2)} \rangle$

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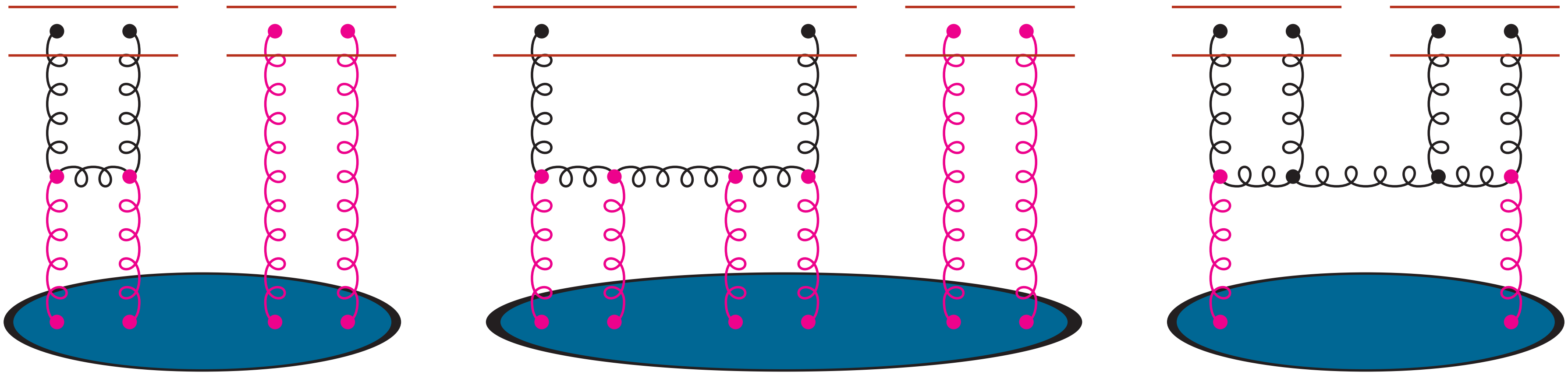
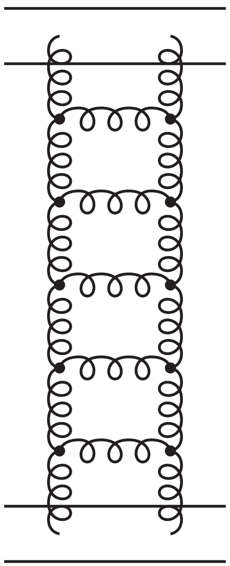


- Usual BFKL ladder $\partial_Y \langle T^{(2)} \rangle \propto \langle T^{(2)} \rangle$
- fan diagram $\longrightarrow$ saturation effects $\partial_Y \langle T^{(2)} \rangle \propto \langle T^{(3)} \rangle$
- splitting $\longrightarrow$ fluctuations, pomeron loops $\partial_Y \langle T^{(2)} \rangle \propto \langle T \rangle$

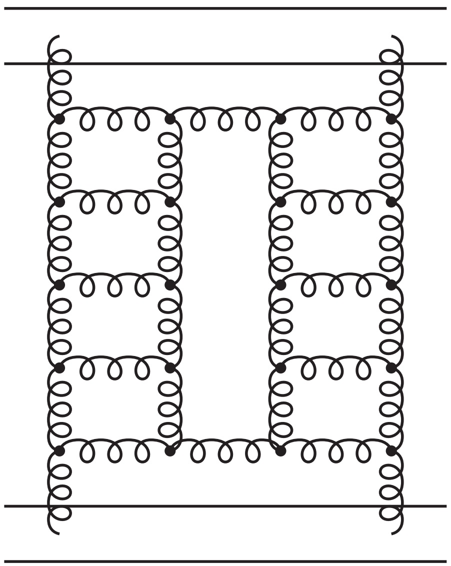
Why fuctuations

Why do we need fluctuations ?

- The fluctuation term acts as a seed for $\langle T^2 \rangle$
- Then, it grows like 2 pomerons !



$\sim e^{\omega Y}$



$\sim \alpha_s^2 e^{2\omega Y}$

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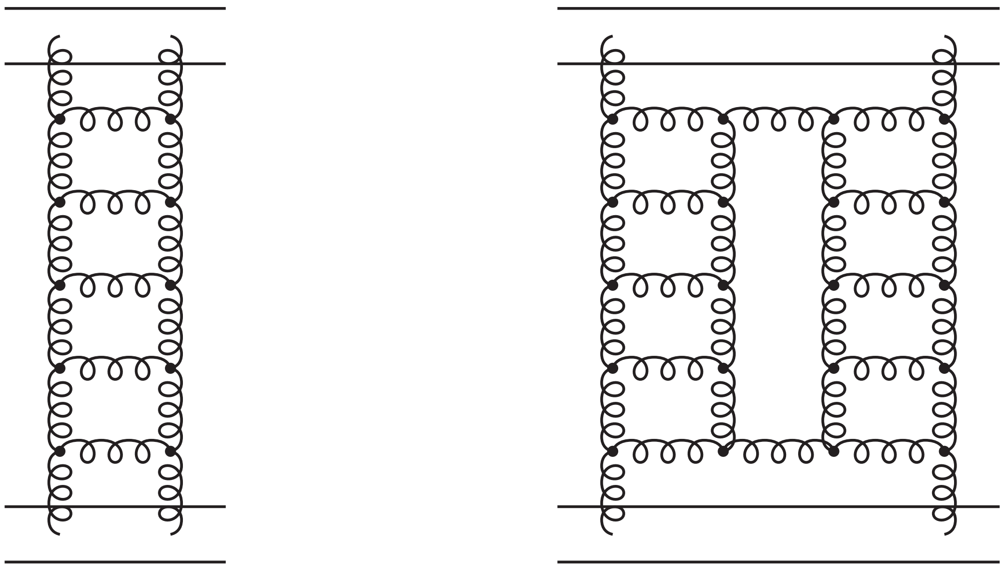


$\sim e^{\omega Y}$ $\sim \alpha_s^2 e^{2\omega Y}$

Comparable for rapidities

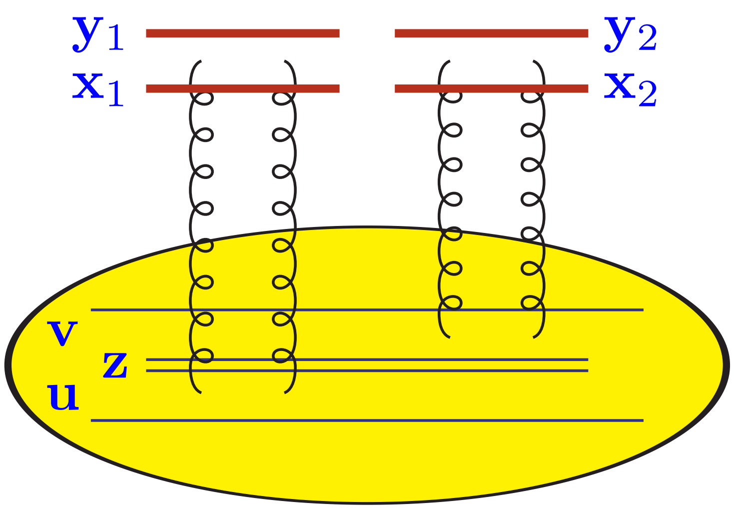
$$Y \sim \frac{1}{\omega_P} \log(1/\alpha_s^2) \sim \frac{1}{\alpha_S} \log(1/\alpha_s^2)$$

$\Rightarrow$ need to include everything.

Computation of the fluctuation term

Expected to be important for dilute target

$\Rightarrow$ Target made of dipoles



$$\underbrace{\int_{\mathbf{uvz}} \frac{\bar{\alpha}}{2\pi} \frac{(\mathbf{u}-\mathbf{v})^2}{(\mathbf{u}-\mathbf{z})^2(\mathbf{z}-\mathbf{v})^2}}_{\text{BFKL splitting}} \underbrace{\alpha_s^2 \mathcal{A}_0(\mathbf{x}_1\mathbf{y}_1|\mathbf{uz})\, \alpha_s^2 \mathcal{A}_0(\mathbf{x}_2\mathbf{y}_2|\mathbf{zv})}_{\text{interaction (2GE)}} \underbrace{\frac{1}{\alpha_s^2} \nabla^2_{\mathbf{u}} \nabla^2_{\mathbf{v}} \langle T_{\mathbf{uv}} \rangle}_{\text{dipole density}}$$

Evolution hierarchy

$\Rightarrow$ complicated hierarchy

$$\partial_Y \left\langle T^{(2)}(\mathbf{x}_1,\mathbf{y}_1;\mathbf{x}_2,\mathbf{y}_2)\right\rangle$$

$$= \frac{\bar\alpha}{2\pi}\int d^2z\,\frac{(\mathbf{x}_2-\mathbf{y}_2)^2}{(\mathbf{x}_2-\mathbf{z})^2(\mathbf{z}-\mathbf{y}_2)^2}\Big[\left\langle T^{(2)}(\mathbf{x}_1,\mathbf{y}_1;\mathbf{x}_2,\mathbf{z})\right\rangle + \left\langle T^{(2)}(\mathbf{x}_1,\mathbf{y}_1;\mathbf{z},\mathbf{y}_2)\right\rangle$$

$$- \left\langle T^{(2)}(\mathbf{x}_1,\mathbf{y}_1;\mathbf{x}_2,\mathbf{y}_2)\right\rangle - \left\langle T^{(3)}(\mathbf{x}_1,\mathbf{y}_1;\mathbf{x}_2,\mathbf{z};\mathbf{z},\mathbf{y}_2)\right\rangle + (1\leftrightarrow 2)\Big]$$

$$+ \frac{\bar\alpha}{2\pi}\left(\frac{\alpha_s}{2\pi}\right)^2\int_{\mathbf{uvz}} \mathcal{M}_{\mathbf{uvz}}\mathcal{A}_0(\mathbf{x}_1\mathbf{y}_1|\mathbf{uz})\mathcal{A}_0(\mathbf{x}_2\mathbf{y}_2|\mathbf{zv})\,\nabla^2_{\mathbf{u}}\nabla^2_{\mathbf{v}}\left\langle T^{(1)}(\mathbf{u},\mathbf{v})\right\rangle$$

- Saturation: important when $T^{(2)}\sim T^{(1)}\sim 1$ i.e. near unitarity
- Fluctuations: important when $T^{(2)}\sim \alpha_s^2 T^{(1)}$ or $T\sim\alpha_s^2$ i.e. dilute regime

Langevin equation

Infinite hierarchy $\equiv$ Langevin equation

$$\begin{aligned}\partial_Y T_{\mathbf{xy}} &= \frac{\bar\alpha}{2\pi}\int_{\mathbf{z}} \mathcal{M}_{\mathbf{xyz}}\left[T_{\mathbf{xz}}+T_{\mathbf{zy}}-T_{\mathbf{xy}}-T_{\mathbf{xz}}T_{\mathbf{zy}}\right]\\ &+ \frac{1}{2}\frac{\bar\alpha}{2\pi}\frac{\alpha_s}{2\pi}\int_{\mathbf{uvz}} \mathcal{A}_0(\mathbf{xy}|\mathbf{uz})\frac{|\mathbf{u}-\mathbf{v}|}{(\mathbf{u}-\mathbf{z})^2}\sqrt{\nabla^2_{\mathbf{u}}\nabla^2_{\mathbf{v}}T_{\mathbf{uv}}}\,\nu(\mathbf{u},\mathbf{v},\mathbf{z};Y)\end{aligned}$$

where ν is a Gaussian white noise

$$\begin{aligned}\langle\nu(\mathbf{u},\mathbf{v},\mathbf{z};Y)\rangle &= 0\\ \langle\nu(\mathbf{u},\mathbf{v},\mathbf{z};Y)\nu(\mathbf{u}',\mathbf{v}',\mathbf{z}';Y')\rangle &= \delta(\bar\alpha Y-\bar\alpha Y')\delta^{(2)}(\mathbf{u}-\mathbf{v}')\delta^{(2)}(\mathbf{z}-\mathbf{z}')\delta^{(2)}(\mathbf{v}-\mathbf{u}')\end{aligned}$$

- Hierarchy obtained by averaging events with different realization of the noise
- problem: non-local & off-diagonal noise !

Langevin equation

Simple example of noise term: zero space dimension

$$\partial_t u(t) = \sqrt{2\kappa u}\,\nu(t) \qquad \text{with } \langle \nu(t)\nu(t')\rangle = \delta(t-t')$$

$$\text{discrete } t \overset{\text{Itô}}{\Rightarrow} \quad u(t_j+\delta t) = u(t_j) + \delta t\,\sqrt{2\kappa u}\,\nu_j \qquad \langle \nu_i\nu_j\rangle = \frac{1}{\delta t}\delta_{ij}$$

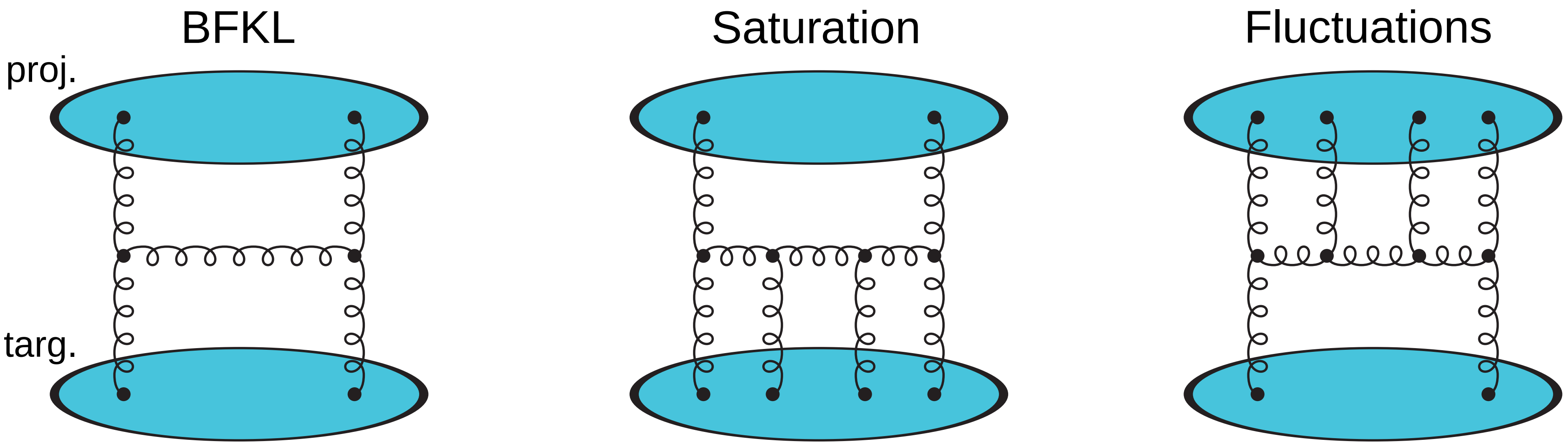
$$\Rightarrow \quad F(u_{j+1}) = F(u_j) + \delta t\sqrt{2\kappa u_j}\nu_j F'(u_j) + \delta t^2\kappa u_j \nu_j^2 F''(u_j)$$

$$\Rightarrow \quad \partial_t \langle F(u)\rangle = \kappa \langle u F''(u)\rangle$$

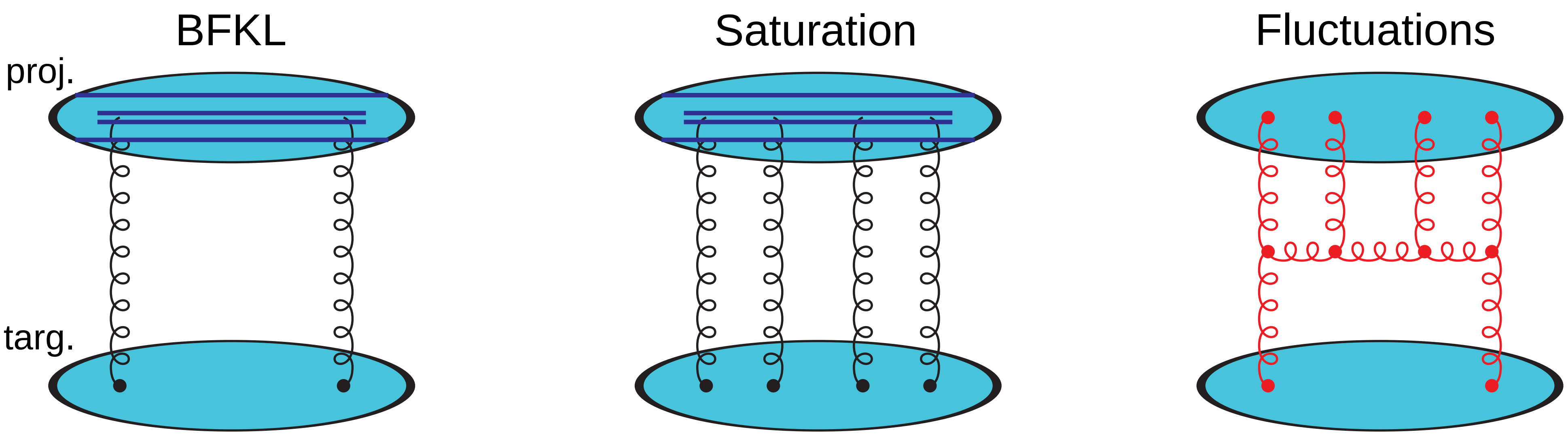
$$F(u) = u^n \quad \Rightarrow \quad \partial_t \langle u^n\rangle = n(n-1)\kappa \left\langle u^{n-1}\right\rangle$$

corresponding to the fluctuation term.

Target vs. projectile



duality: target-proectile symmetry $\equiv$ boost invariance



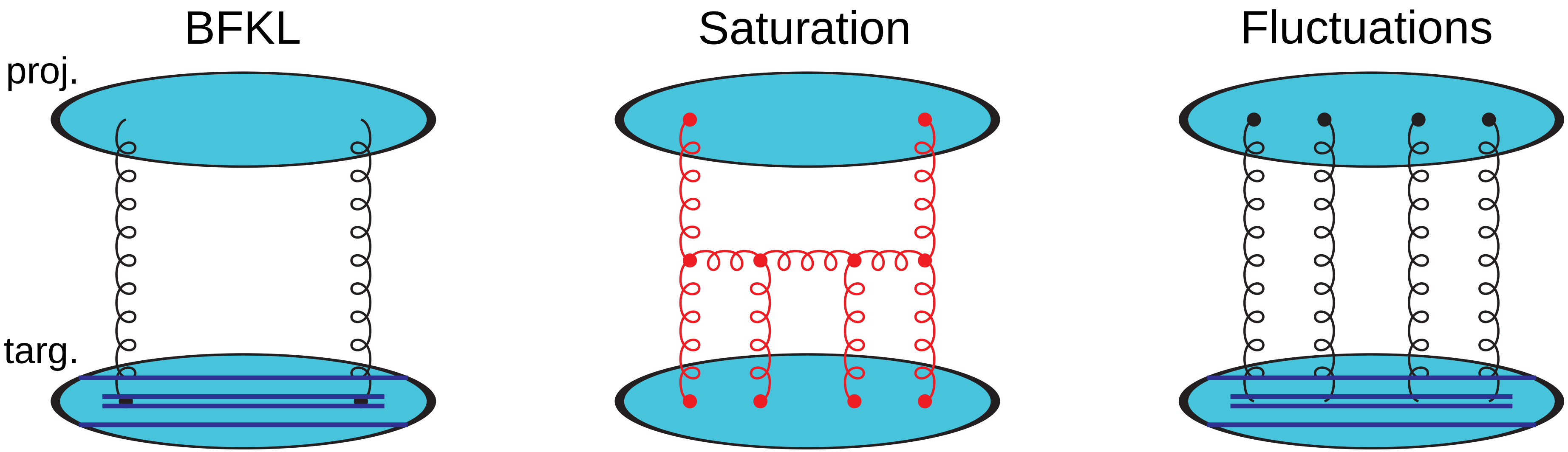
<u>duality</u>: target-proectile symmetry $\equiv$ boost invariance

Projectile wavefunction evolution

BFKL & saturation from dipole splitting

fluctuations $\equiv$ gluon merging (or recombination, saturation)

Target vs. projectile



duality: target-proectile symmetry $\equiv$ boost invariance

Projectile wavefunction evolution

BFKL & saturation from dipole splitting

fluctuations $\equiv$ gluon merging (or recombination, saturation)

Target wavefunction evolution

BFKL & fluctuations from dipole splitting

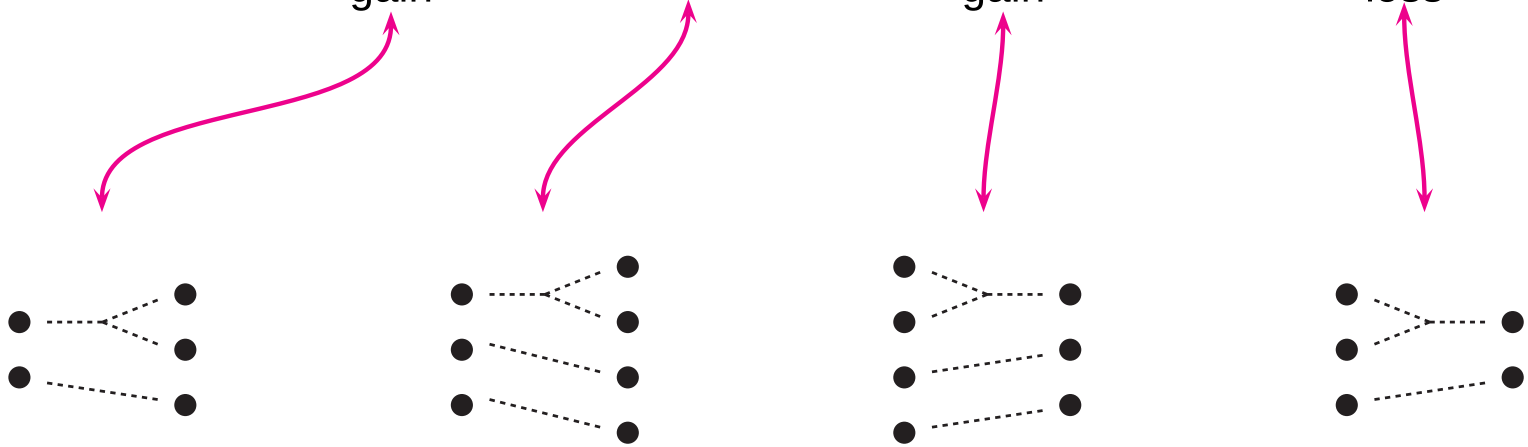
saturation $\equiv$ multiple scatterings

Reaction-diffusion

<u>Reaction-diffusion process</u> $A \underset{\sigma}{\overset{\gamma}{\rightleftharpoons}} A + A$

Master equation: $P_n \equiv$ proba to have n particles

$$\partial_t P_n = \underbrace{\gamma\,(n-1)P_{n-1}}_{\text{gain}} - \underbrace{\gamma\, n P_n}_{\text{loss}} + \underbrace{\sigma\, n(n+1)P_{n+1}}_{\text{gain}} - \underbrace{\sigma\, n(n-1)P_n}_{\text{loss}}$$



Particle densities: we observe a subset of k particles

$$\langle n^k \rangle \equiv \sum_{N=k}^{\infty} \frac{N!}{(N-k)!} P_N$$

Reaction-diffusion process $A \underset{\sigma}{\overset{\gamma}{\rightleftharpoons}} A + A$

Master equation: $P_n \equiv$ proba to have n particles

$$\partial_t P_n = \underbrace{\gamma\,(n-1)P_{n-1}}_{\text{gain}} - \underbrace{\gamma\, n P_n}_{\text{loss}} + \underbrace{\sigma\, n(n+1)P_{n+1}}_{\text{gain}} - \underbrace{\sigma\, n(n-1)P_n}_{\text{loss}}$$

Evolution equation: $\langle n^k \rangle \equiv$ particle density/correlators

$$\partial_t \langle n^k \rangle = \gamma\, k \langle n^k \rangle + \gamma\, k(k-1) \langle n^{k-1} \rangle - \sigma\, k(k+1) \langle n^{k+1} \rangle$$

Scattering amplitude for this system off a target

$$\mathcal{A}(t) = \sum_{k=0}^{\infty} (-)^k \, \langle n^k \rangle_{t_0} \, \langle T^k \rangle_{t-t_0}$$

t_0-independent $\Rightarrow$

$$\partial_t \langle T^k \rangle = \underbrace{\gamma \langle T^k \rangle}_{\text{BFKL}} - \underbrace{\gamma \langle T^{k+1} \rangle}_{\text{sat.}} + \underbrace{\sigma \langle T^{k-1} \rangle}_{\text{fluct.}}$$

For QCD particle = (effective) dipoles

Dipole plitting $\equiv$ BFKL kernel

$$\gamma \sim \bar\alpha \frac{(\mathbf{x}-\mathbf{y})^2}{(\mathbf{x}-\mathbf{z})^2(\mathbf{z}-\mathbf{y})^2}$$

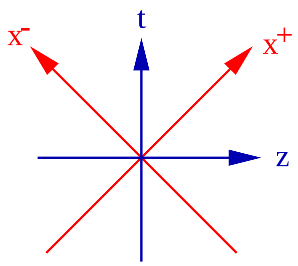
Effective dipole merging

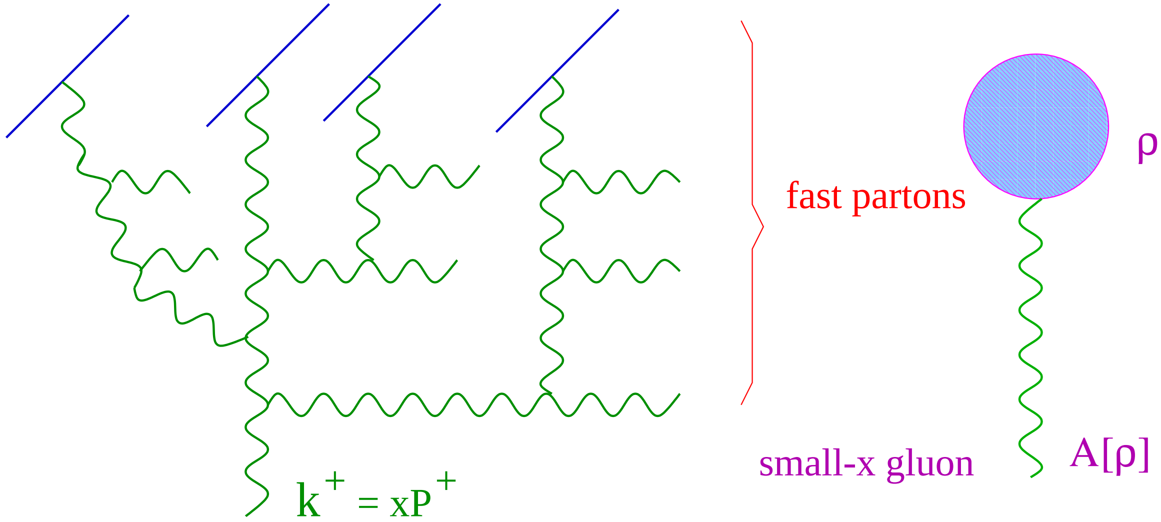
$$\sigma(\mathbf{x}_1\mathbf{y}_1,\mathbf{x}_2\mathbf{y}_2 \to \mathbf{u}\mathbf{v})$$

$$\sim \bar\alpha\alpha_s^2\nabla_{\mathbf{u}}^2\nabla_{\mathbf{v}}^2\left\{\mathcal{M}_{\mathbf{uvz}}\log^2\left[\frac{(\mathbf{x}_1-\mathbf{u})^2(\mathbf{y}_1-\mathbf{z})^2}{(\mathbf{x}_1-\mathbf{z})^2(\mathbf{y}_1-\mathbf{u})^2}\right]\log^2\left[\frac{(\mathbf{x}_2-\mathbf{v})^2(\mathbf{y}_2-\mathbf{z})^2}{(\mathbf{x}_2-\mathbf{z})^2(\mathbf{y}_2-\mathbf{v})^2}\right]\right\}$$

Remarks:

- merging not always positive
- fluctuations = gluon-number fluctuations
- Can be obtained from projectile or target point of view
- Known at large N_c.

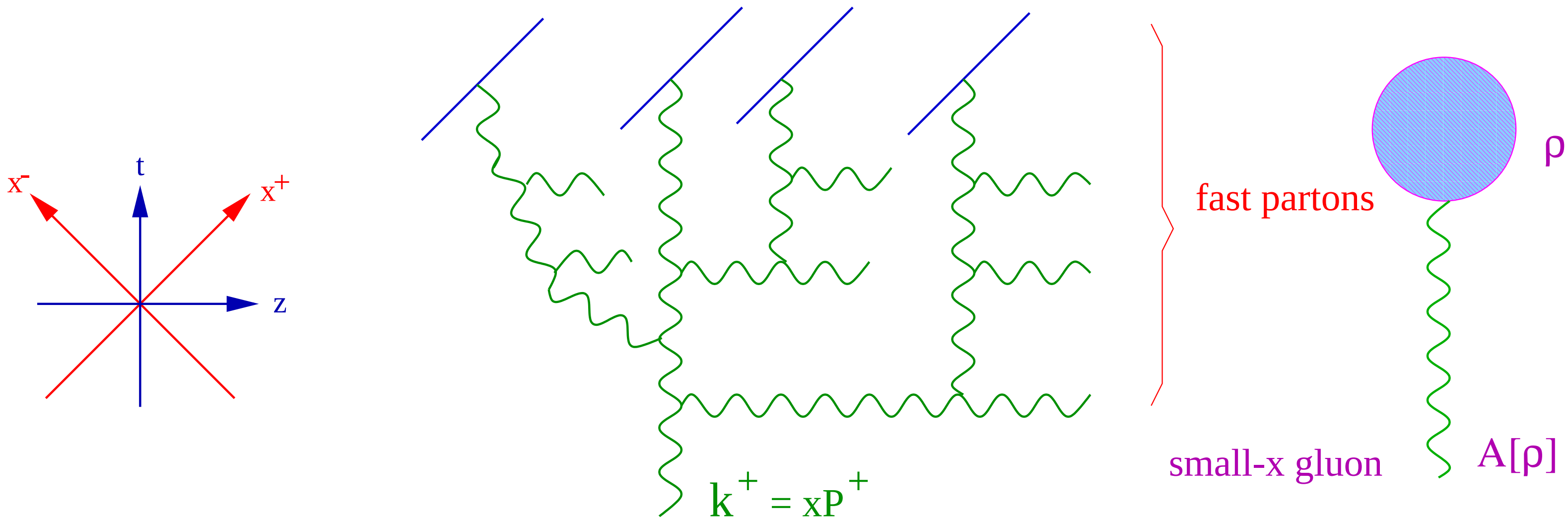




Effective theory for High-Energy QCD:

- Theory for the **gluonic field**: Color
- Small-x $\equiv$ classical field radiated by frozen fast gluons
  Large-x $\equiv$ random distribution of color sources: Glass
- Large occupation number: Condensate

Equation for the probability distribution of the color charge $W_Y[\rho]$



- fast gluons: frozen, source for slow partons

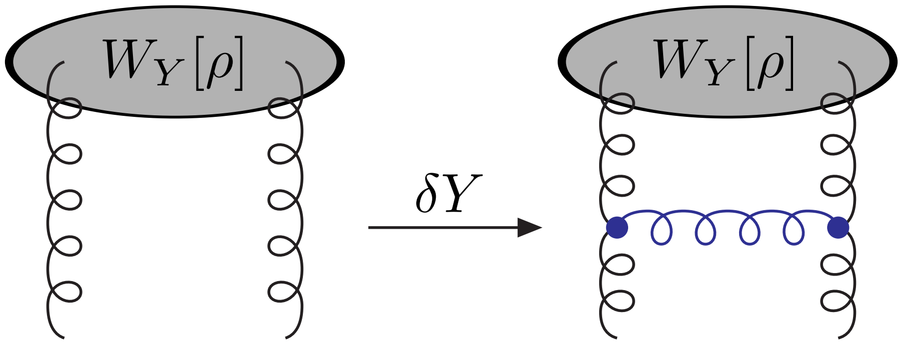
$$(D_\mu F^{\mu\nu})_a = \delta^{\nu +}\rho_a(x^-, \mathbf{x}_\perp)$$

- Random source: correlators computed using the probability distribution $W_Y[\rho]$

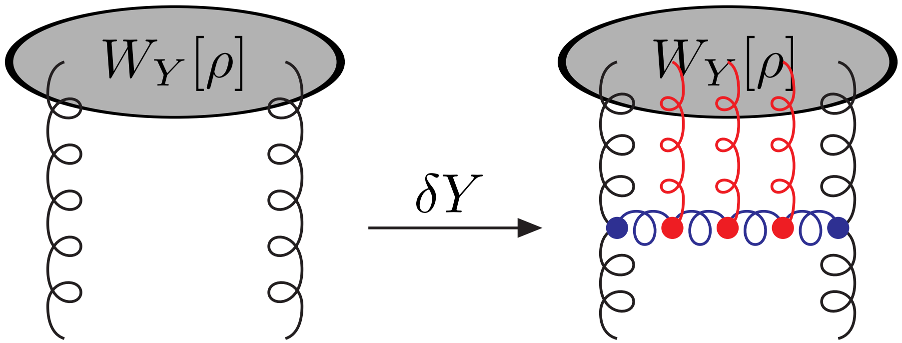
$$\langle A_a^i A_a^i \rangle = \int \mathcal{D}\rho\, W_Y[\rho]\, A_a^i A_a^i$$

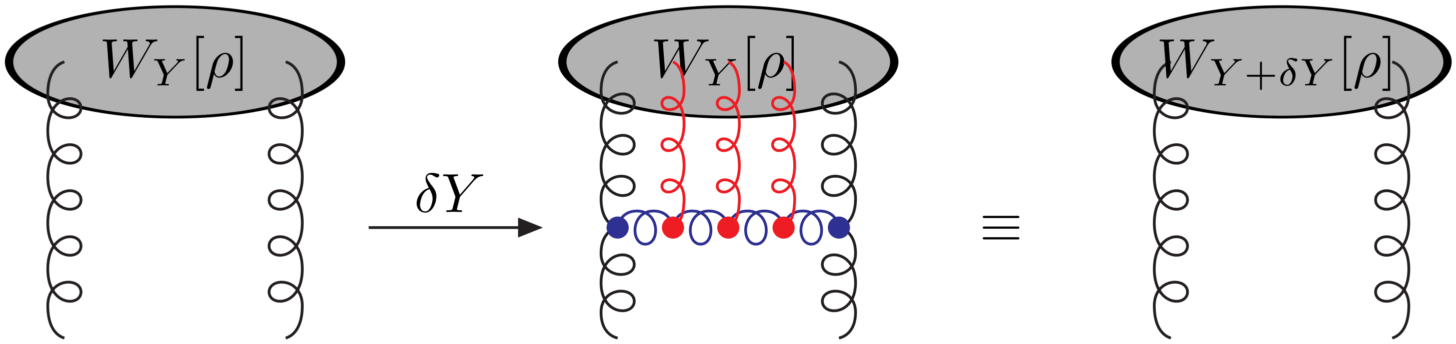
- Strong field $A \sim 1/g$ (equivalent to $n \sim 1/\alpha_s$ or $T \sim 1$)

Color Glass Condensate (3/4)



Color Glass Condensate (3/4)





Evolution

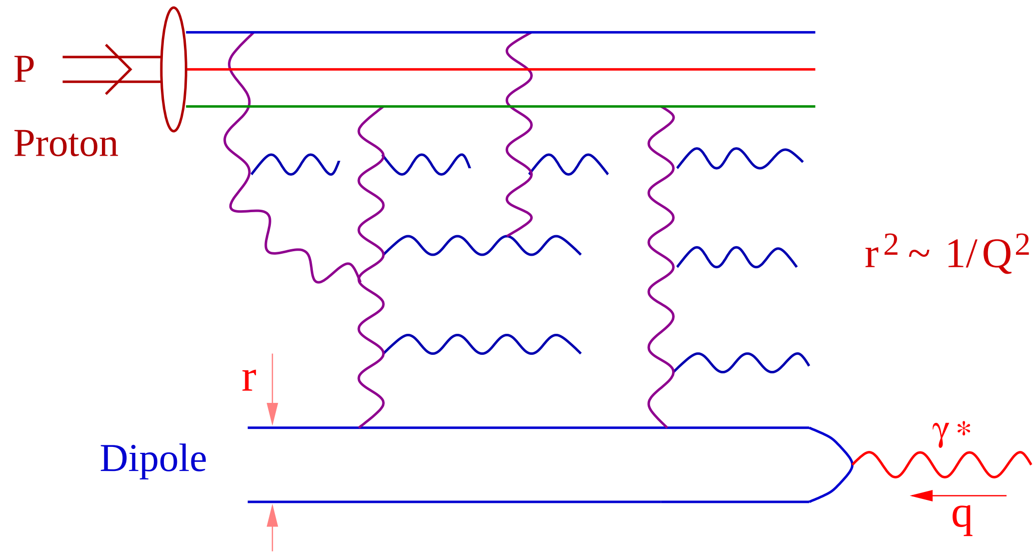
$$\partial_Y W_Y[\rho] = \frac{1}{2}\int_{\mathbf{xy}} \frac{\delta}{\delta\rho^a_{\mathbf{x}}}\chi^{ab}_{\mathbf{xy}}[\rho]\frac{\delta}{\delta\rho^a_{\mathbf{y}}} W_Y[\rho]$$

[Jalilan-Marian,Iancu,McLerran,Weigert,Leonidov,Kovner,97-00]

Wilson Line

$$V_x = P\,\exp\left[ig\int dx^-\,A^+(x^-,\mathbf{x}_\perp)\right]$$

In the weak field limit $\longrightarrow$ BFKL.

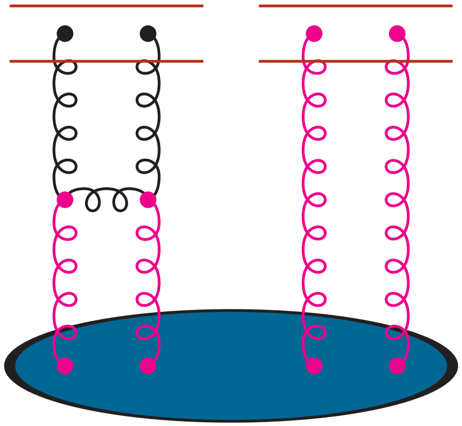


S-matrix: $\gamma^* \to q\bar{q} \to V^\dagger_{\mathbf{x}} V_{\mathbf{y}}$

$$S_Y = \int \mathcal{D}A^+ \, W_Y[A] \, \frac{1}{N_c} \text{tr}(V^\dagger_{\mathbf{x}} V_{\mathbf{y}})$$

Wilson Line

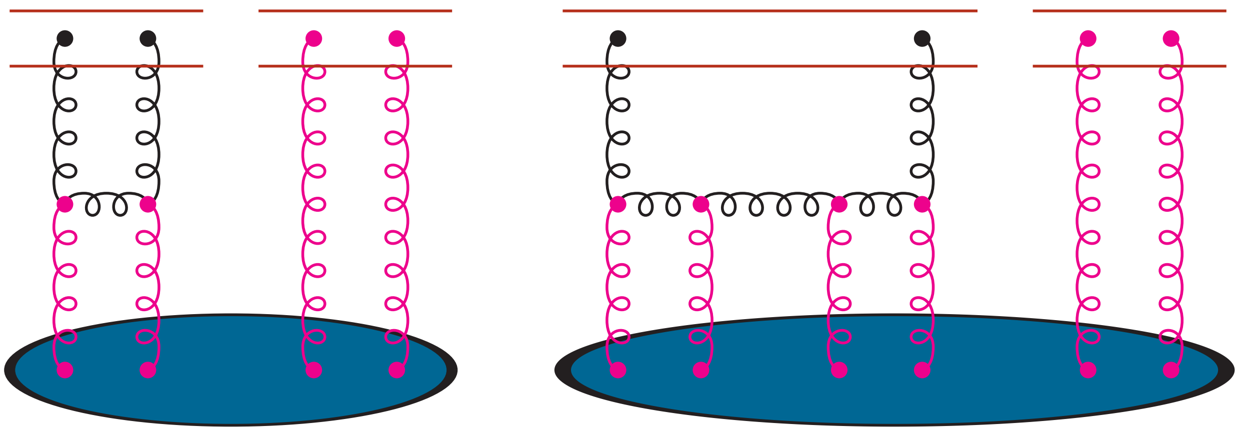
$$V_x = P \, \exp\left[ig \int dx^- \, A^+(x^-, \mathbf{x}_\perp) \right]$$



Ladder-type diagrams $\Rightarrow$ BFKL equation

$$\partial_Y \langle T_{\mathbf{xy}} \rangle = \frac{\bar{\alpha}}{2\pi} \int_{\mathbf{z}} \frac{(\mathbf{x}-\mathbf{y})^2}{(\mathbf{x}-\mathbf{z})^2(\mathbf{z}-\mathbf{y})^2} \left[\langle T_{\mathbf{xz}} \rangle + \langle T_{\mathbf{zy}} \rangle - \langle T_{\mathbf{xy}} \rangle \right]$$

unitarity violations

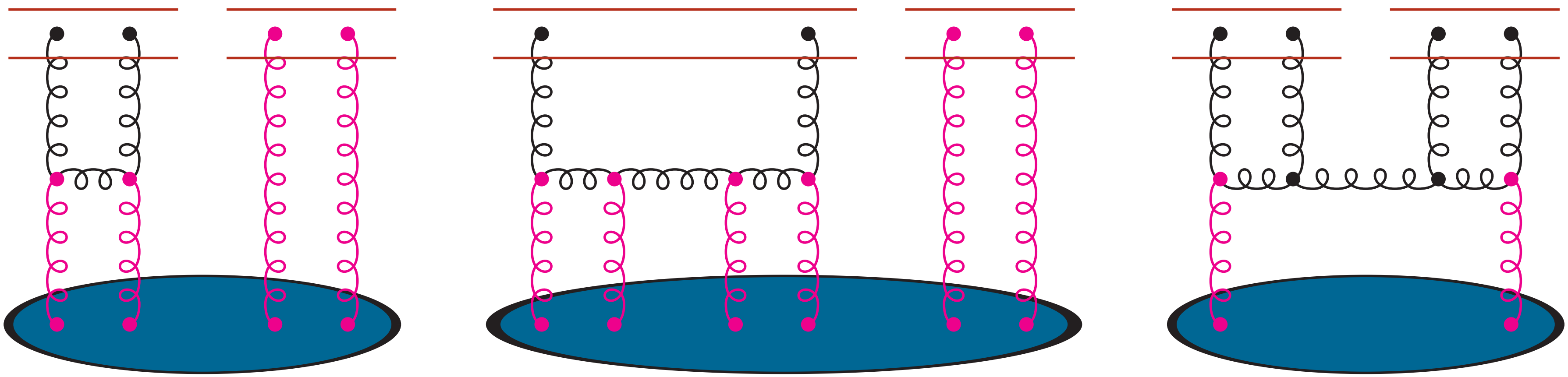


Unitarity corrections: Add fan diagrams ⇒ Balitsky equation

$$\partial_Y \langle T_{\mathbf{xy}} \rangle = \frac{\bar\alpha}{2\pi} \int_{\mathbf{z}} \frac{(\mathbf{x}-\mathbf{y})^2}{(\mathbf{x}-\mathbf{z})^2(\mathbf{z}-\mathbf{y})^2} \left[\langle T_{\mathbf{xz}} \rangle + \langle T_{\mathbf{zy}} \rangle - \langle T_{\mathbf{xy}} \rangle - \langle T_{\mathbf{xz}} T_{\mathbf{zy}} \rangle \right]$$

- infinite hierarchy: Balitsky/JIMWLK
- for $\left\langle T^{(2)}_{\mathbf{xz},\mathbf{zy}} \right\rangle = \langle T_{\mathbf{xz}} \rangle \langle T_{\mathbf{zy}} \rangle$: BK equation

Summary



gluon-number fluctuations: add fluctuations in the target

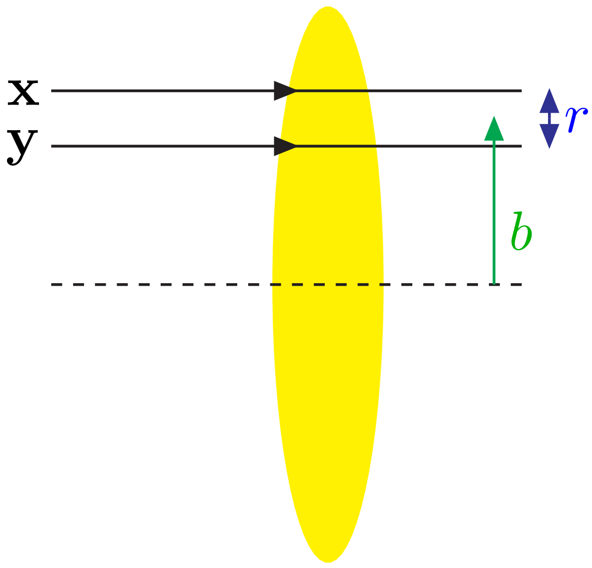
$$\partial_Y \left\langle T^{(2)}_{\mathbf{x}_1\mathbf{y}_1,\mathbf{x}_2\mathbf{y}_2} \right\rangle\Big|_{\text{flucu}} = \frac{1}{2}\frac{\bar{\alpha}}{2\pi}\left(\frac{\alpha_s}{2\pi}\right)^2 \int_{\mathbf{uvz}} \mathcal{M}_{\mathbf{uvz}} \mathcal{A}_0(1|\mathbf{uz})\mathcal{A}_0(2|\mathbf{zv})\nabla^2_{\mathbf{u}}\nabla^2_{\mathbf{v}} \langle T_{\mathbf{uv}} \rangle$$

- equivalent to a reaction-diffusion problem
- projectile-target duality

Asymptotic solutions for scattering amplitudes

- *impact-parameter-independent BK*
- *BK at nonzero momentum transfer*
- *including fluctuations*

[Munier, Peschanski,05]



$$T(\mathbf{x},\mathbf{y}) \longrightarrow T(\mathbf{r};\mathbf{b}) \longrightarrow T(\mathbf{r})$$

Momentum space:

$$T(\mathbf{k}) = \frac{1}{2\pi}\int \frac{d^2r}{r^2} e^{i\mathbf{r}.\mathbf{k}} T(\mathbf{r}) = \int \frac{dr^2}{r^2} J_0(kr) T(r)$$

BK equation

$$\partial_Y T(k) = \frac{\bar{\alpha}}{\pi}\int \frac{dp^2}{p^2}\left[\frac{p^2T(p)-k^2T(k)}{|k^2-p^2|} + \frac{k^2T(k)}{\sqrt{4p^4+k^4}}\right] - \bar{\alpha}T^2(k)$$

Link with statistical physics (1/2)

[S. Munier, R. Peschanski]

b-independent situation: momentum space ($L = \log(k^2/k_0^2)$)

$$\partial_{\bar\alpha Y} T(k) = \chi_{\text{BFKL}}(-\partial_L) T(k) - T^2(k)$$

Diffusive approximation:

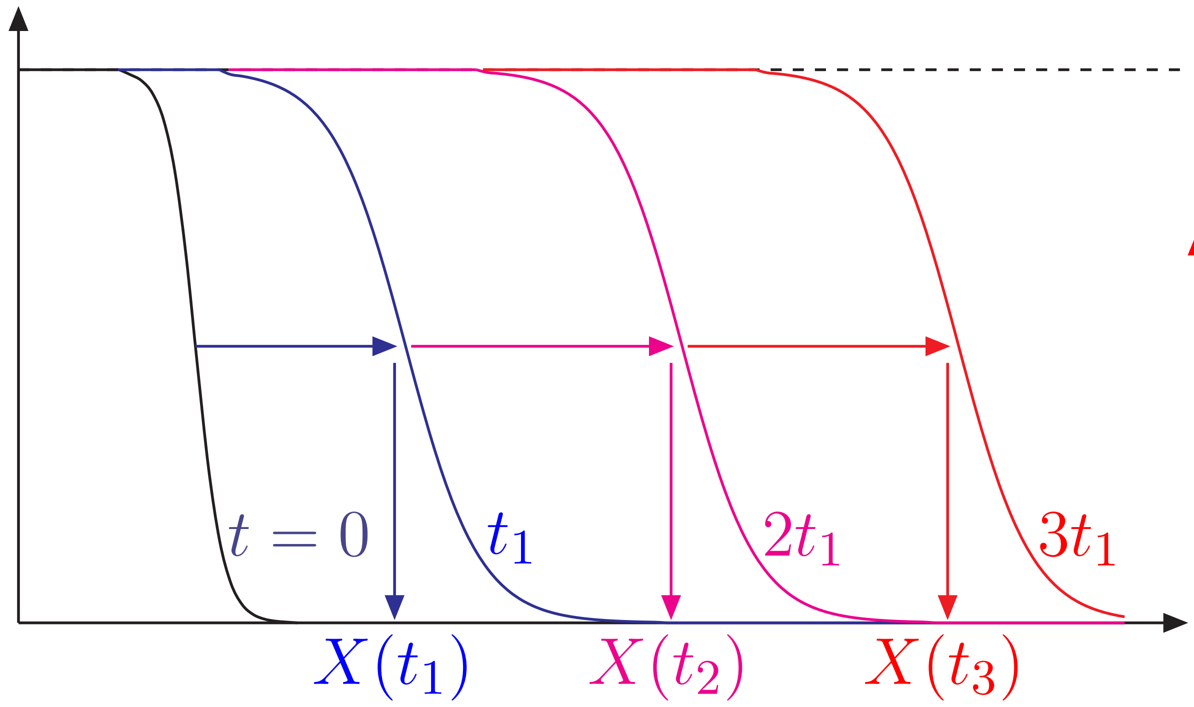
$$\chi_{\text{BFKL}}(-\partial_L) = \chi(\tfrac{1}{2}) + \tfrac{1}{2}(\partial_L + \tfrac{1}{2})^2$$

Time $t = \bar\alpha Y$, Space $x \approx \log(k^2)$, $u \propto T$

$$\partial_t u(x,t) = \partial_x^2 u(x,t) + u(x,t) - u^2(x,t)$$

Fisher-Kolmogorov-Petrovsky-Piscounov (F-KPP)

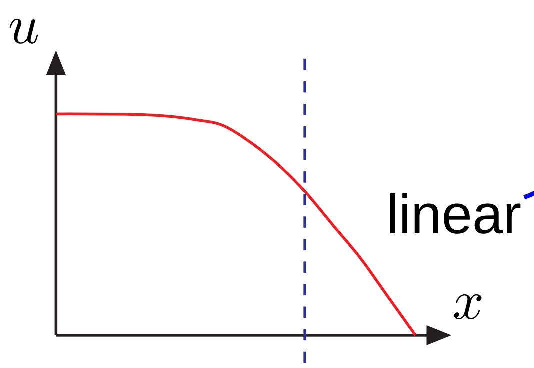
Link with statistical physics (2/2)



Asymptotic solution:
travelling wave

$$u(x,t) = u(x - v_c t)$$

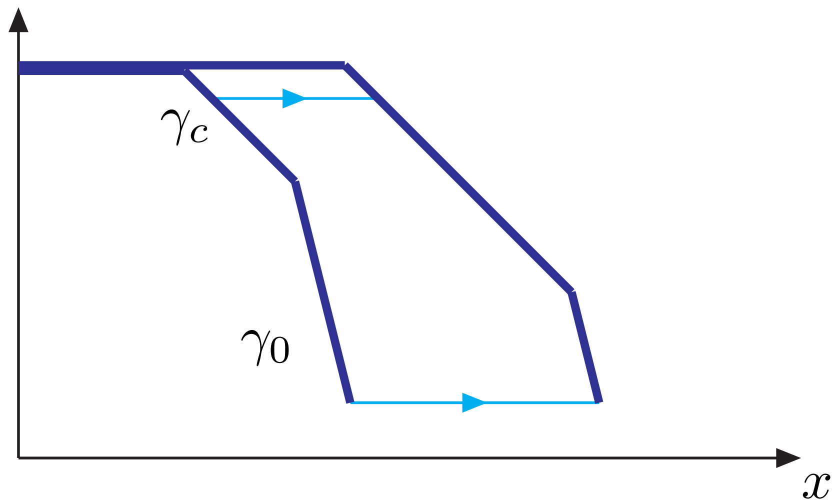
Position: $X(t) = X_0 + v_c t$

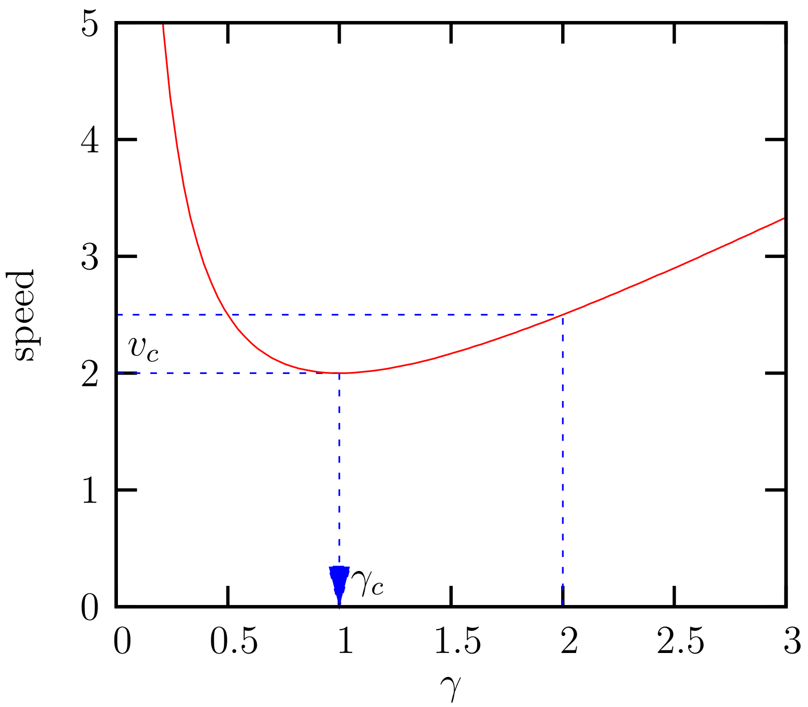


$$\partial_t u = \partial_x^2 u + u = \omega(-\partial_x) u$$

with $\omega(\gamma) = \gamma^2 + 1$

$$\Rightarrow u(x,t) = \int \frac{d\gamma}{2i\pi}\, u_0(\gamma)\, \exp[\omega(\gamma)t - \gamma x]$$





The minimal speed is selected during evoution

More generaly, it is true under the conditions:

- 0 is an unstable solution, 1 is a stable solution
- The initial condition is steep enough
- The linear equation admits solution of the form

$$T_{\text{lin}} = \int_{c-i\infty}^{c+\infty} \frac{d\gamma}{2i\pi} a_0(\gamma) \exp\left[\omega(\gamma)Y - \gamma L\right]$$

$\Rightarrow$ Travelling waves with critical speed

$v_c = \text{min}\frac{\omega(\gamma)}{\gamma}$ or $v_c = \frac{\omega(\gamma_c)}{\gamma_c} = \omega'(\gamma_c)$

Asymptotic behaviour (1/2)

More generaly, it is true under the conditions:

- 0 is an unstable solution, 1 is a stable solution
  BFKL growth, BK damping
- The initial condition is steep enough
  Colour transparency
- The linear equation admits solution of the form

$$T_{\text{lin}} = \int_{c-i\infty}^{c+\infty} \frac{d\gamma}{2i\pi} a_0(\gamma) \exp\left[\omega(\gamma)Y - \gamma L\right]$$

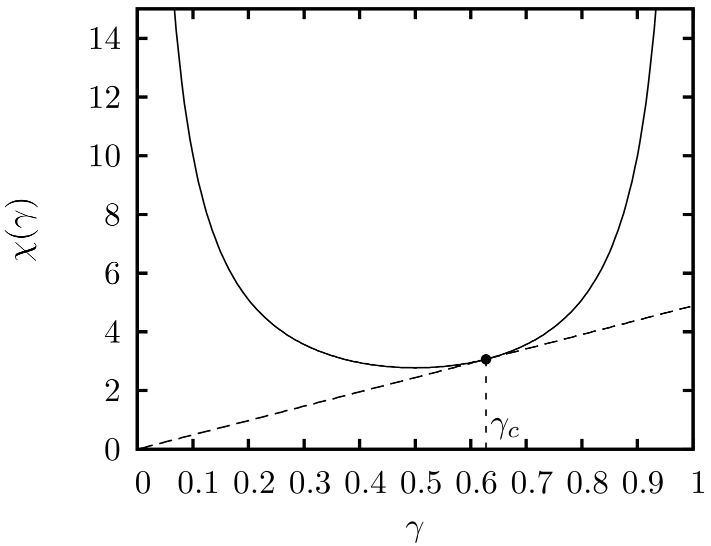
BFKL: $\omega(\gamma) = \bar{\alpha}\chi(\gamma) = \bar{\alpha}\ \left[2\psi(1) - \psi(\gamma) - \psi(1-\gamma)\right]$

$\Rightarrow$ Travelling waves with critical speed

$v_c = \min \frac{\omega(\gamma)}{\gamma}$ or $v_c = \frac{\omega(\gamma_c)}{\gamma_c} = \omega'(\gamma_c)$

BK equation: linear part $\equiv$ BFKL kernel

$$\gamma_c = 0.6275$$
$$v_c = 4.8834\bar{\alpha}$$



Tail of the front:

$$T(k,Y) = T\left(\frac{k^2}{Q_s^2(Y)}\right) \approx \log\left(\frac{k^2}{Q_s^2(Y)}\right)\left|\frac{k^2}{Q_s^2(Y)}\right|^{-\gamma_c} \exp\left(-\frac{\log^2(k^2/Q_s^2(Y))}{\bar{\alpha}\chi''(\gamma_c)Y}\right)$$

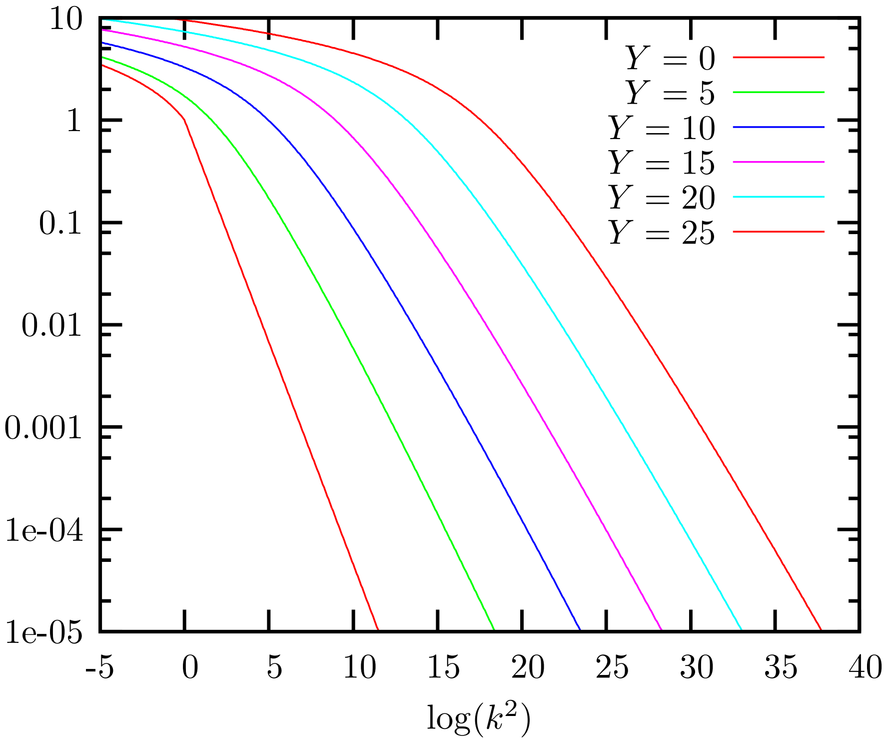
Geometric scaling

Saturation Scale:

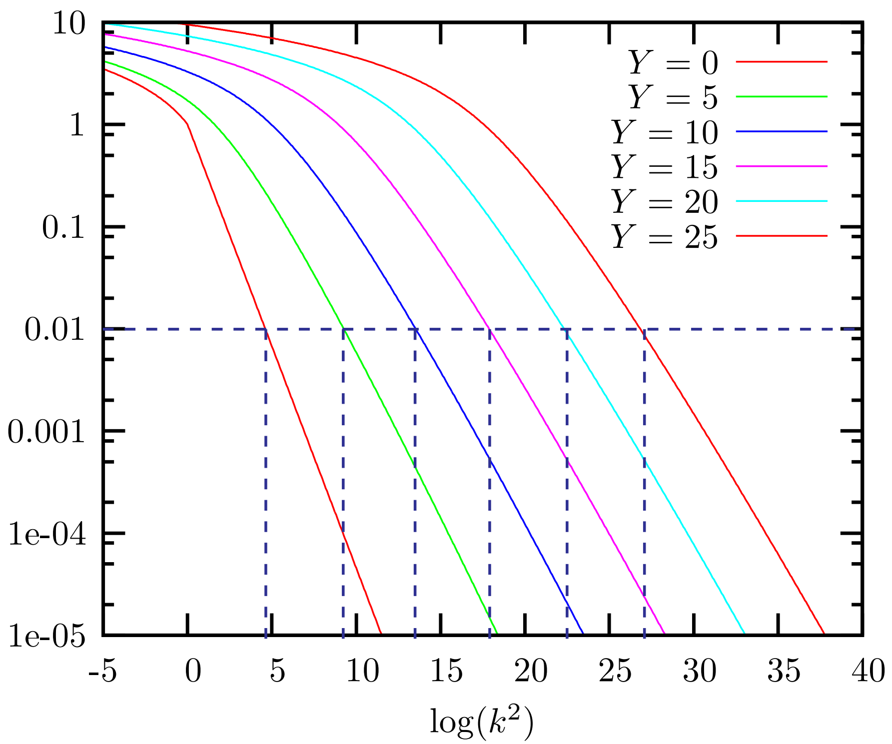
$$\log(Q_s^2(Y)) = v_c Y - \frac{3}{2\gamma_c}\log(Y) - \frac{3}{\gamma_c^2}\sqrt{\frac{2\pi}{\bar{\alpha}\chi''(\gamma_c)}}\frac{1}{\sqrt{Y}}$$

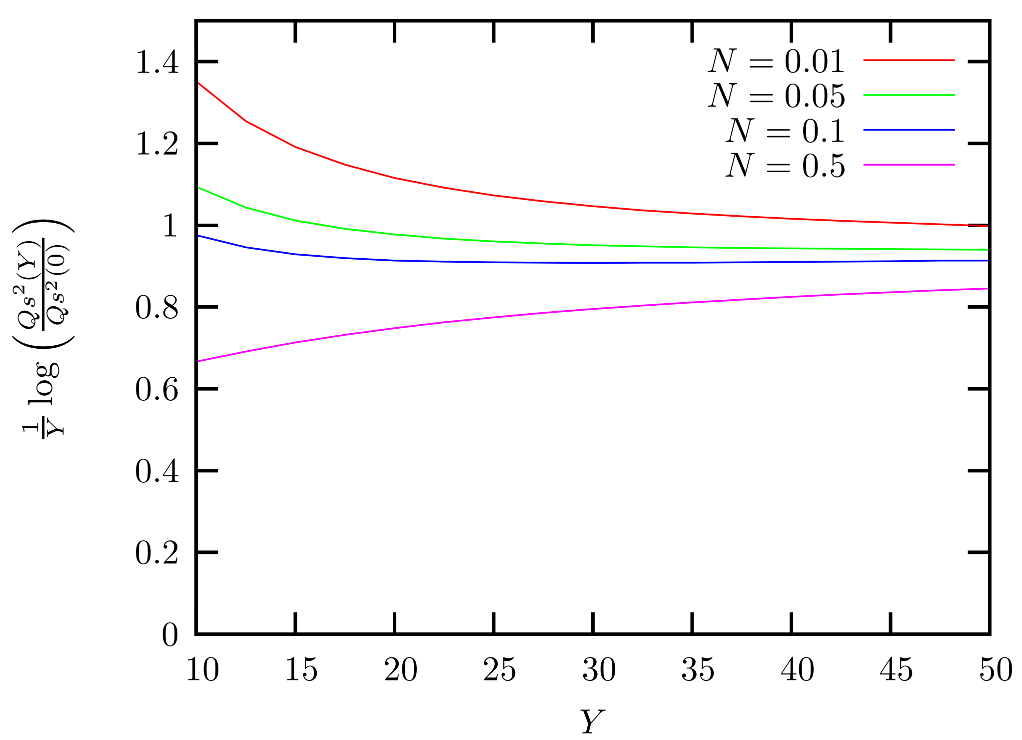
Geometric scaling

Numerical simulations: $\bar{\alpha} = 0.2$



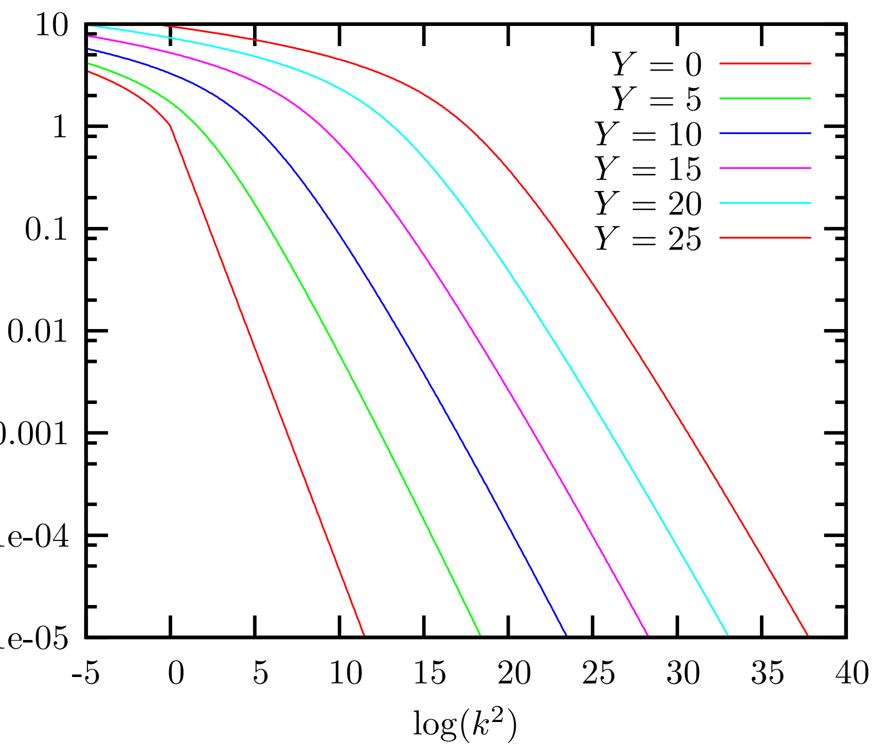
Numerical simulations: $\bar{\alpha} = 0.2$

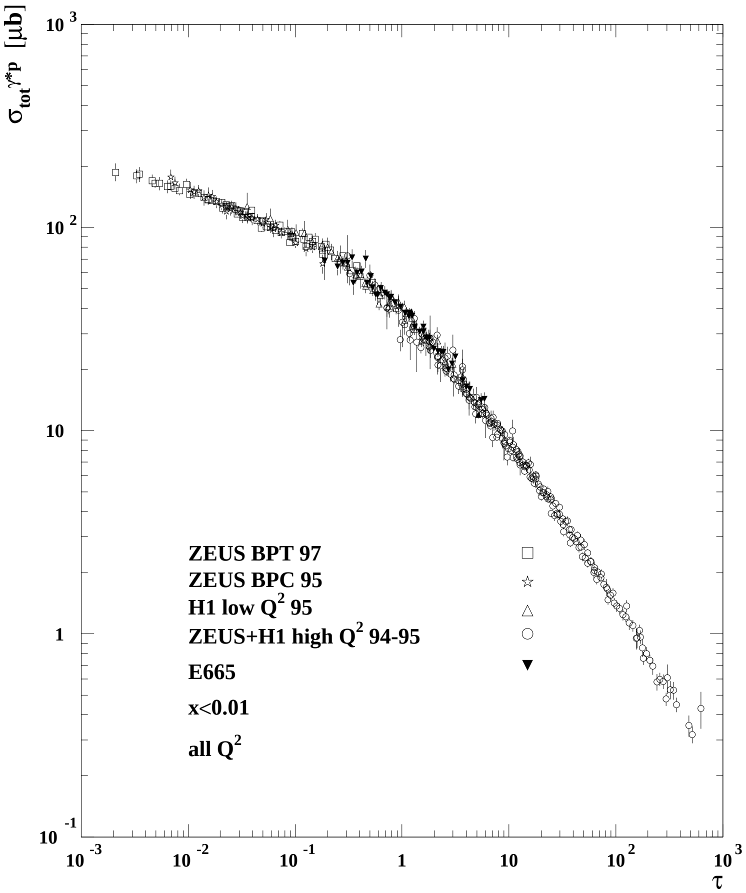




[Kwiecinski, Stasto; 01]

Observed in the HERA data for F_2^p





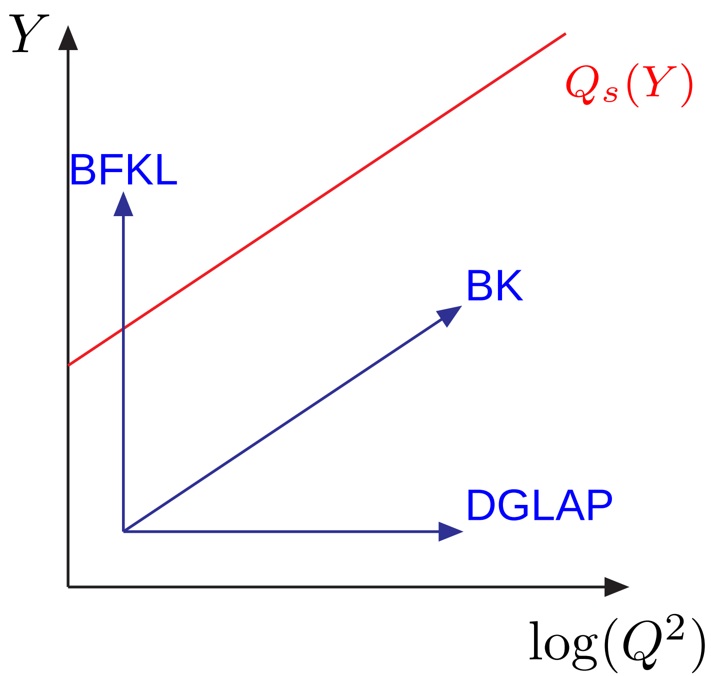
Another approach

[Mueller, Triantafyllopoulos, 02]

Idea:

saturation cuts $Q^2 < Q_s^2$

$\Rightarrow$ BK $\equiv$ BFKL + boundary



BFKL solution:

$$T = \int \frac{d\gamma}{2i\pi} T_0(\gamma) \exp\left[\bar\alpha\chi(\gamma)Y - \gamma\log(Q^2/\mu^2)\right]$$

Conditions:

- Saddle point: $\bar\alpha\chi'(\gamma_c)Y - \log(Q_s^2(Y)/\mu^2) = 0$
- Barrier: $\bar\alpha\chi(\gamma_c)Y - \gamma_c\log(Q_s^2(Y)/\mu^2) = 0$

$\Rightarrow \gamma_c\chi'(\gamma_c) = \chi(\gamma)$ and $\log(Q_s^2/\mu^2) = \bar\alpha\chi'(\gamma_c)Y$.

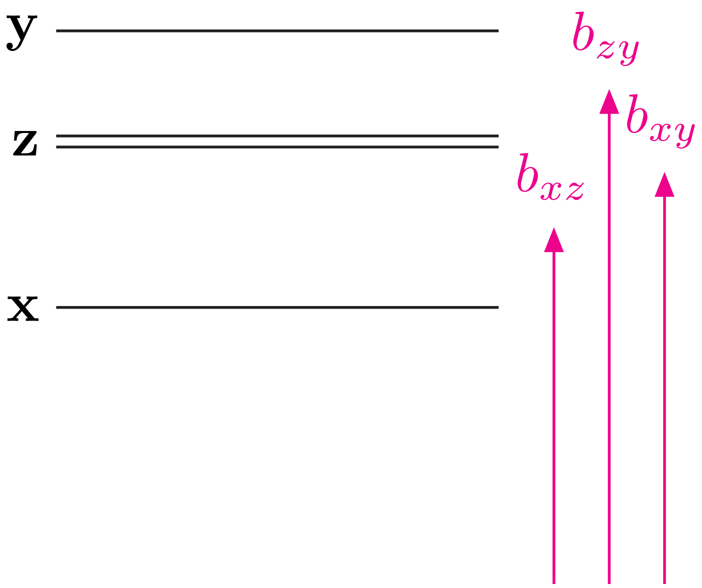
Asymptotic solutions

The full BK equation

Question: do we have the same properties for the full BK equation ?

$$\partial_Y \langle T_{\mathbf{xy}} \rangle = \frac{\bar{\alpha}}{2\pi} \int d^2z \, \mathcal{M}_{\mathbf{xyz}} \left[\langle T_{\mathbf{xz}} \rangle + \langle T_{\mathbf{zy}} \rangle - \langle T_{\mathbf{xy}} \rangle - \langle T_{\mathbf{xz}} \rangle \langle T_{\mathbf{zy}} \rangle \right]$$

Problem:



Diffusion/non-locality in b

New form of the BK equation

[C. Marquet, R. Peschanski, G.S., 05]

Solution: go to momentum space

$$\tilde{T}(\mathbf{k},\mathbf{q}) = \int d^2x\, d^2y\, e^{i\mathbf{k}.\mathbf{x}} e^{i(\mathbf{q}-\mathbf{k}).\mathbf{y}} \frac{T(\mathbf{x},\mathbf{y})}{(\mathbf{x}-\mathbf{y})^2}$$

new form of the BK equation

$$\begin{aligned}\partial_Y \tilde{T}(\mathbf{k},\mathbf{q}) &= \frac{\bar{\alpha}}{\pi}\int \frac{d^2k'}{(k-k')^2}\left\{\tilde{T}(\mathbf{k}',\mathbf{q}) - \frac{1}{4}\left[\frac{k^2}{k'^2} + \frac{(q-k)^2}{(q-k')^2}\right]\tilde{T}(\mathbf{k},\mathbf{q})\right\} \\ &- \frac{\bar{\alpha}}{2\pi}\int d^2k'\, \tilde{T}(\mathbf{k},\mathbf{k}')\tilde{T}(\mathbf{k}-\mathbf{k}',\mathbf{q}-\mathbf{k}')\end{aligned}$$

- locality in q for the BFKL term
- Asymptotic solutions: study the linear kernel
  We need a superposition of waves

Solutions of the full BFKL kernel:

$$\tilde{T}_{\text{lin}}(\mathbf{k},\mathbf{q}) = \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \frac{d\gamma}{2i\pi}\, e^{\bar{\alpha}\chi(\gamma)Y} f^{\gamma}(\mathbf{k},\mathbf{q})\phi_0(\gamma,\mathbf{q})$$

with

$$f^{\gamma}(\mathbf{k},\mathbf{q}) = \frac{\Gamma^2(\gamma)}{\Gamma^2\left(\frac{1}{2}+\gamma\right)} \frac{2}{|k|} \left|\frac{q}{4k}\right|^{2\gamma-1} \underbrace{{}_2\mathrm{F}_1\left(\gamma,\gamma;2\gamma;\frac{q}{k}\right) {}_2\mathrm{F}_1\left(\gamma,\gamma;2\gamma;\frac{\bar{q}}{\bar{k}}\right)}_{\longrightarrow 1 \quad \text{when } k \gg q} - (\gamma \to 1-\gamma)$$

$$\Rightarrow \quad \tilde{T}_{\text{lin}}(\mathbf{k},\mathbf{q}) = \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \frac{d\gamma}{2i\pi}\, \phi_0(\gamma,\mathbf{q}) \exp\left[\bar{\alpha}\chi(\gamma)Y - \gamma\log\left(\frac{k^2}{q^2}\right)\right]$$

⇒ geometric scaling for the full BK equation

Saturation Scale: same Y dependence as previously

$$Q_s^2(Y) \sim q^2 \exp\left[v_c Y - \frac{3}{2\gamma_c}\log(Y) - \frac{3}{\gamma_c^2}\sqrt{\frac{2\pi}{\bar{\alpha}\chi''(\gamma_c)}}\frac{1}{\sqrt{Y}}\right]$$
$$\sim q^2\Omega_s^2(Y)$$

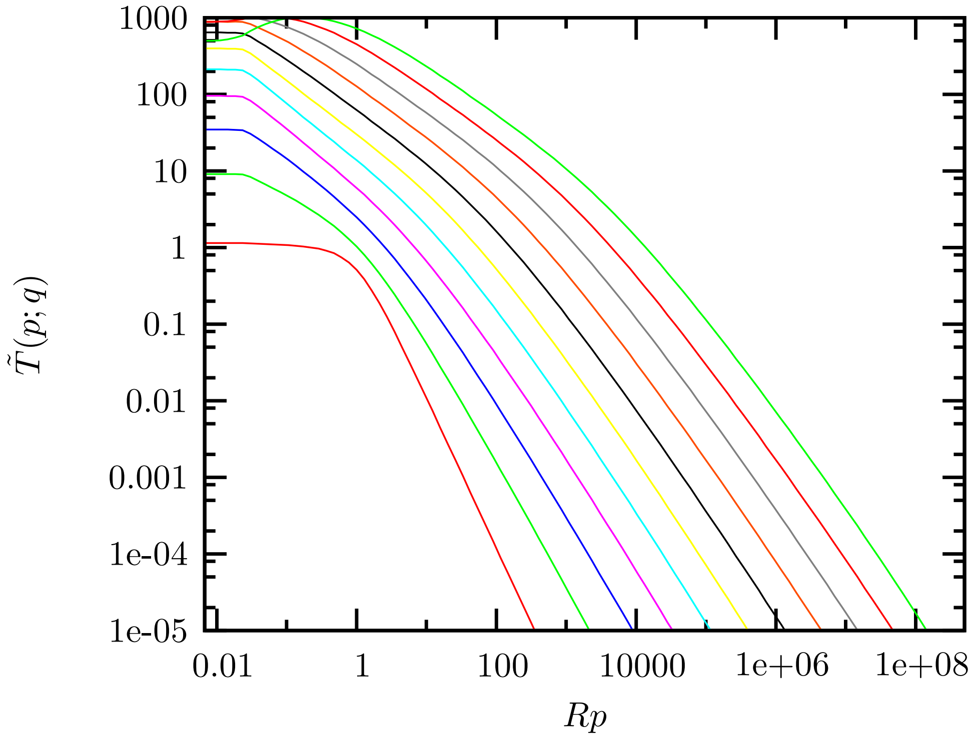
Tail of the front: same slope γ_c

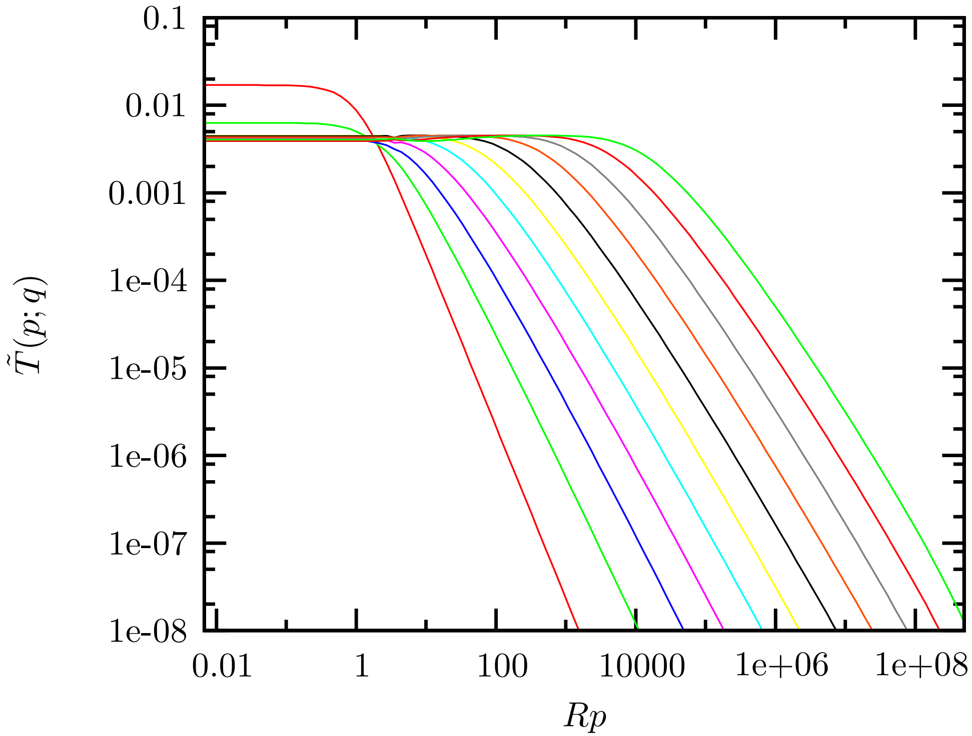
$$T(k,Y) = T\left(\frac{k^2}{q^2\Omega_s^2(Y)}\right) \approx \log\left(\frac{k^2}{q^2\Omega_s^2(Y)}\right)\left|\frac{k^2}{q^2\Omega_s^2(Y)}\right|^{-\gamma_c}$$

Note: more careful treatment gives $Q_s^2 = \Omega^2(Y)\,\mathrm{max}(q^2, Q_T^2)$

Predicts geometric scaling for t-dependent processes

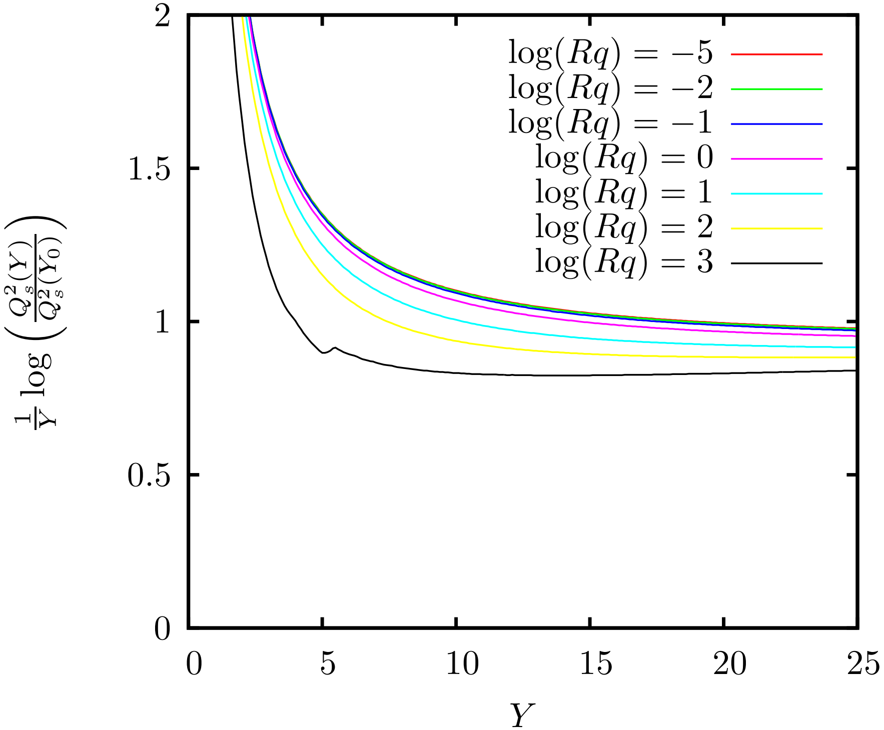
Dependence on momentum transfer k: travelling waves

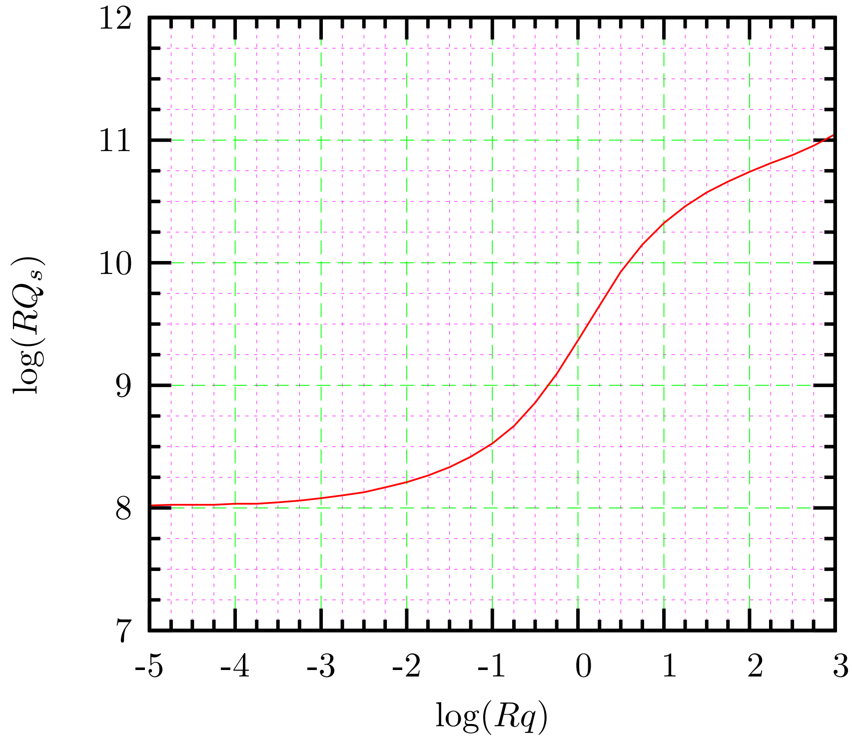




- formation of a travelling wave at large p (or k) $\Rightarrow$ Geometric scaling
- cut-off effect in the infrared region

Saturation scale





- Y dependence: converges to v_c
- q dependence: scales like a constant or linearly ($Y = 25$)

Solutions

Fluctuation effects

[E. Iancu, A. Mueller, S. Munier, 04]

no b-dependence + coarse-graining $\longrightarrow$ Langevin equation

$$\partial_Y T(k,Y) = \bar\alpha K_{\text{BFKL}} \otimes T(k,Y) - \bar\alpha T^2(k,Y) + \bar\alpha\sqrt{\kappa\alpha_s^2 T(k,Y)}\,\nu(k,Y)$$

with $\langle \nu(k,Y)\rangle = 0$

$$\langle \nu(k,Y)\nu(k',Y')\rangle = \frac{1}{\bar\alpha}\delta(Y-Y')\,k\delta(k-k')$$

Diffusive approximation

$$\partial_t u(x,t) = \partial_x^2 u + u - u^2 + \sqrt{2\kappa u}\,\nu(x,t)$$

stochastic F-KPP

Numerical solution of

$$\partial_Y T(k,Y) = \bar{\alpha} K_{\text{BFKL}} \otimes T(k,Y) - \bar{\alpha} T^2(k,Y) + \bar{\alpha}\sqrt{\kappa \alpha_s^2 T(k,Y)}\,\nu(k,Y)$$

Dealing with the noise term: $du = \sqrt{2\kappa u}\nu(t) \Rightarrow \partial_t \langle F(u)\rangle = \kappa \langle u F''(u)\rangle$

Associated probability

$$\langle F(u)\rangle = \int du\, F(u)\, P(u,t) \qquad \overset{\partial_t}{\Rightarrow} \qquad \partial_t P(u,t) = \kappa \partial_u^2 \left[u P(u,t)\right]$$

Including the initial contition $u(t=0)=u_0$, we get

$$P_t(u_0 \to u) \;\equiv\; \text{probability to go from } u_0 \text{ to } u \text{ in a time } t.$$

Define the cumulative probability $F_{u_0,t}(u) = \int_{0^-}^{u} dv P_t(u_0 \to v)$.

Numerical solution

Numerical solution of

$$\partial_Y T(k,Y) = \bar{\alpha} K_{\text{BFKL}} \otimes T(k,Y) - \bar{\alpha} T^2(k,Y) + \bar{\alpha}\sqrt{\kappa \alpha_s^2 T(k,Y)}\,\nu(k,Y)$$

Rapidity step δY:

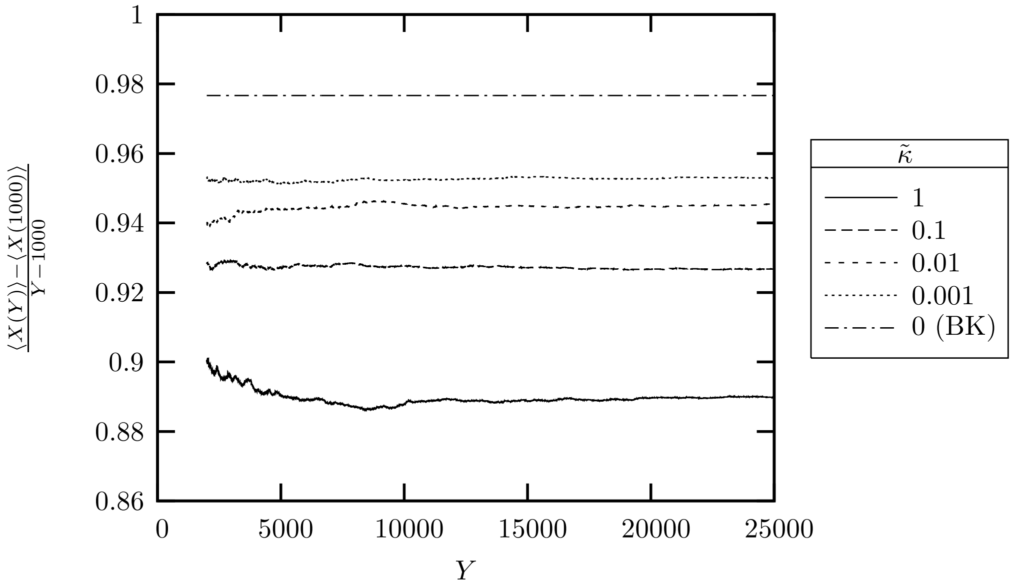
- Step 1: Use probability: $0 < y < 1$ uniform random variable

$$T_{\text{noise}}(k,Y) = F^{-1}_{T(k,Y),\delta Y}(y)$$

- Step 2: Apply the remaining equation

$$T(k,Y+\delta Y) = T_{\text{noise}}(k,Y) + \delta Y \left[\bar{\alpha} K_{\text{BFKL}} \otimes T_{\text{noise}}(k,Y) - \bar{\alpha} T^2_{\text{noise}}(k,Y)\right]$$

[G.S., 05]



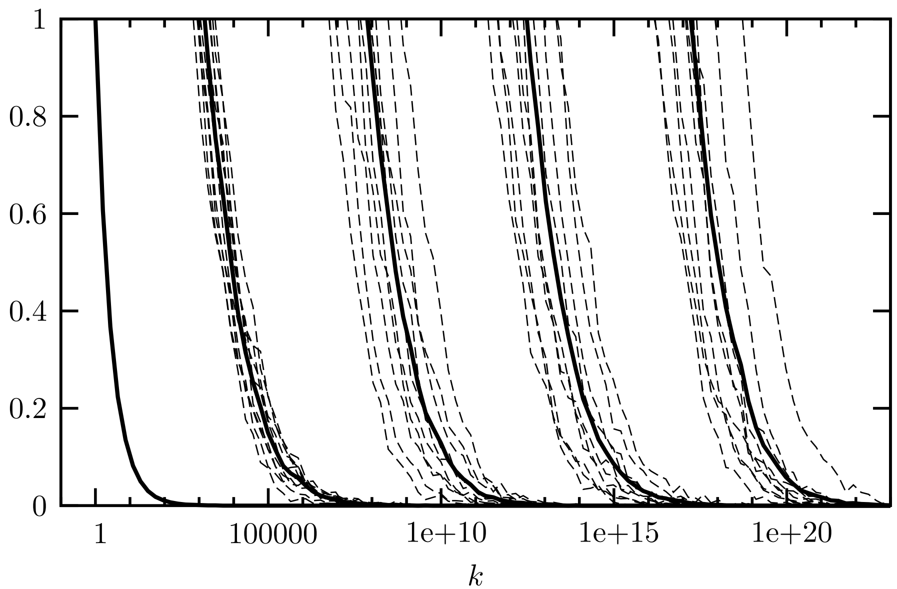
Decrease of the velocity/exponent of the saturation scale

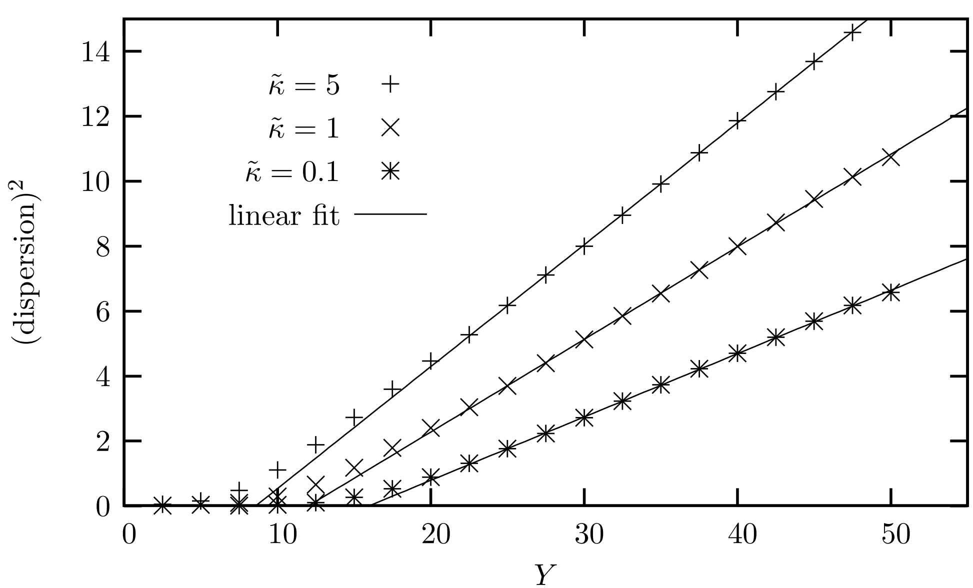
For $\alpha_s \ll 1$ (not true here)[A. Mueller, S. Munier, E. Brunet, B. Derrida], see S. Munier's talk

$$v^* \underset{\alpha_s^2\kappa\to 0}{\longrightarrow} v_{BK} - \frac{\bar{\alpha}\pi^2\gamma_c\chi''(\gamma_c)}{2\log^2(\alpha_s^2\kappa)} + \dots$$

Event properties (2/2)

[G.S., 05]





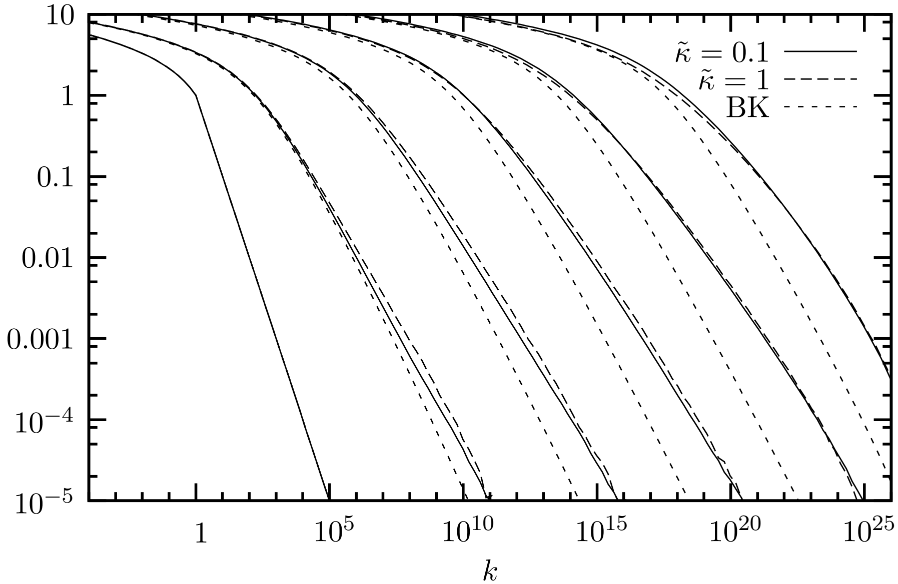
- Dispersion of the events

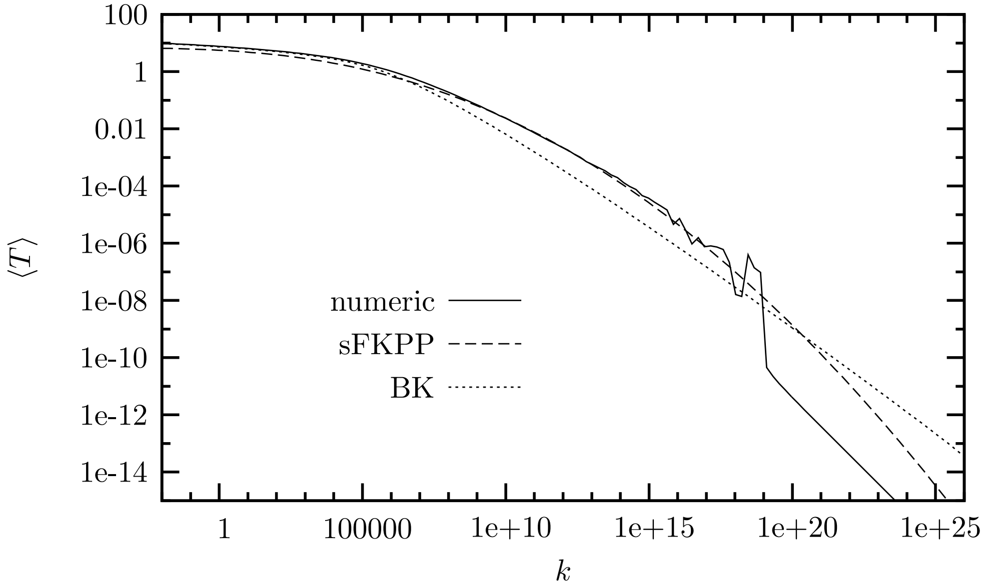
$$\Delta \log[Q_s^2(Y)] \approx \sqrt{D_{\text{diff}}\bar{\alpha} Y} \quad \text{with} \quad D_{\text{diff}} \underset{\alpha_s^2\kappa\to 0}{\sim} \frac{1}{|\log^3(\alpha_s^2\kappa)|}.$$

- No important dispersion in early stages of the evolution !

Averaged amplitude

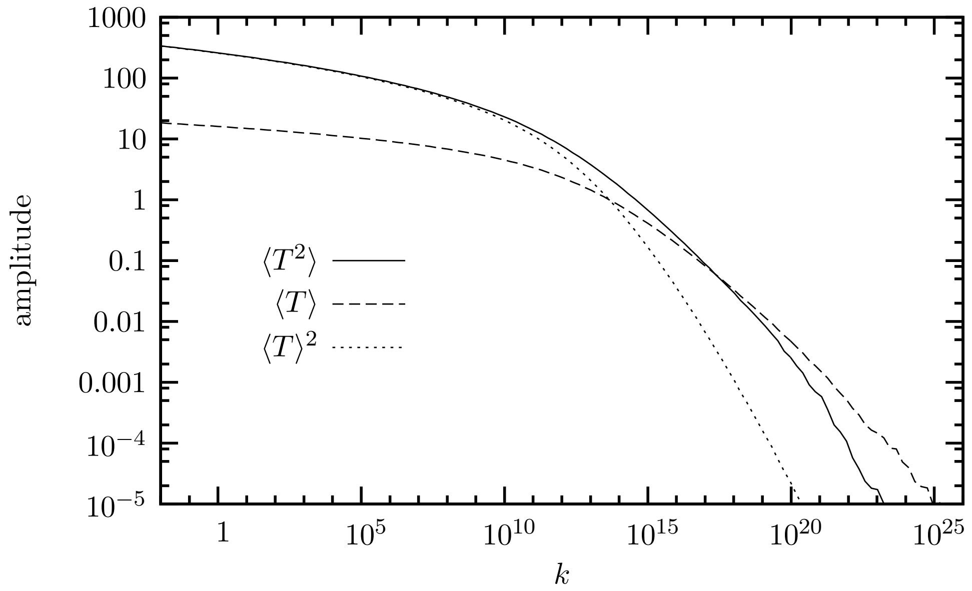
[G.S., 05]





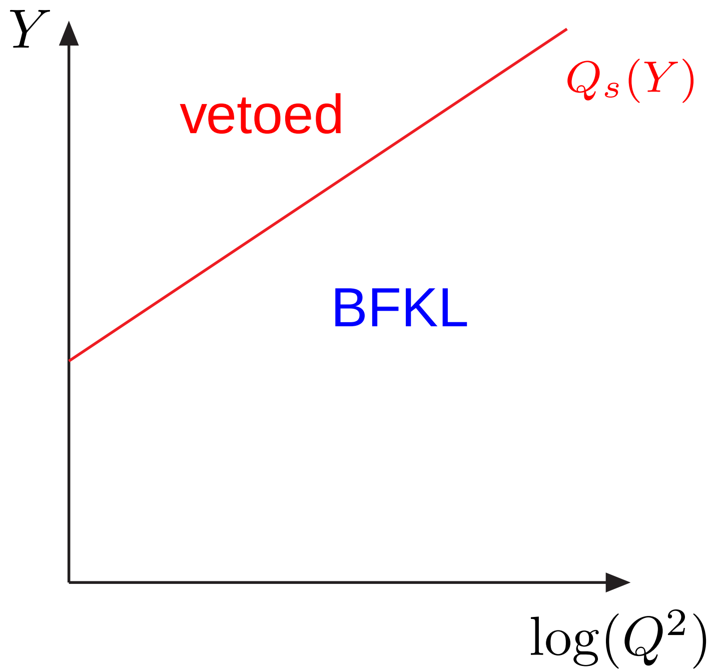
- Clear effect of fluctuations
- Violations of geometric scaling (not in early stages)
- Agrees with predictions

[G.S., 05]



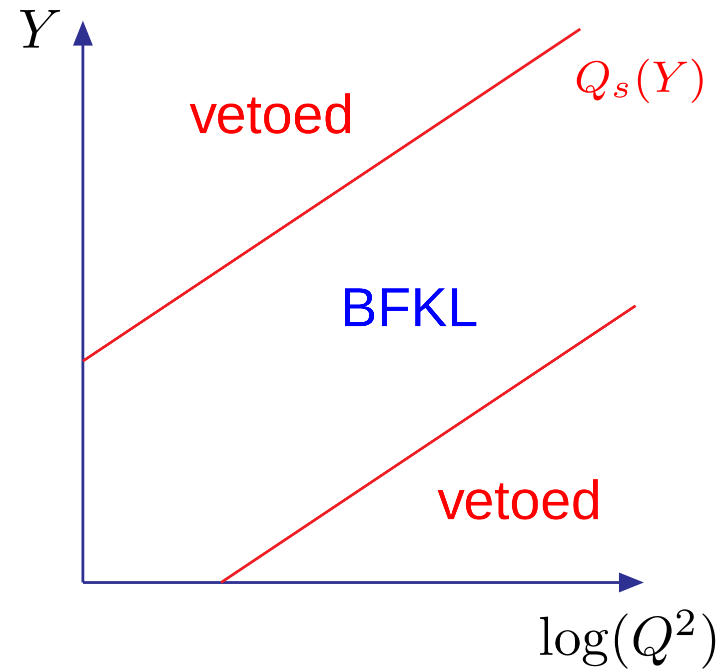
- Dense regime: $\left\langle T^2 \right\rangle \approx \langle T \rangle^2$
- Dilute regime: $\left\langle T^2 \right\rangle \approx \langle T \rangle$ (pre-asymptotics!)

[Mueller, Triantafyllopoulos, 02]



Saturation

[Mueller, Shoshi, 04]

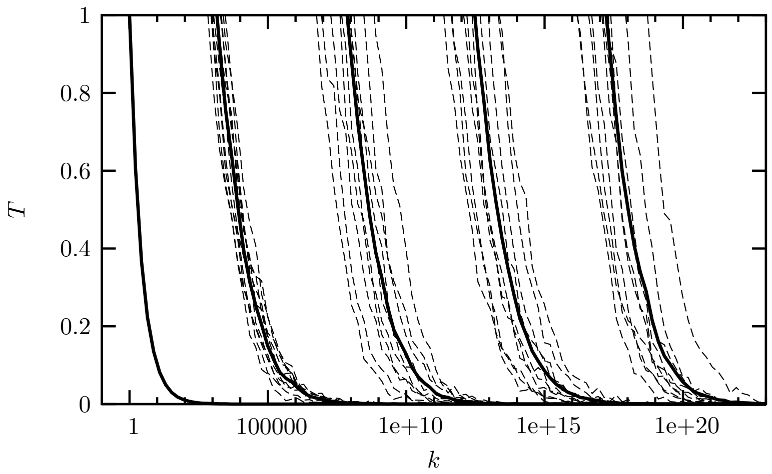


Saturation
+ fluctuations

[Y. Hatta, E. Iancu, C. Marquet, G.S., D. Triantafyllopoulos, 06]

Evolution with saturation & fluctuations $\equiv$

- superposition of unitary front (with geometric scaling)
- with a dispersion (yielding geometric scaling violations)

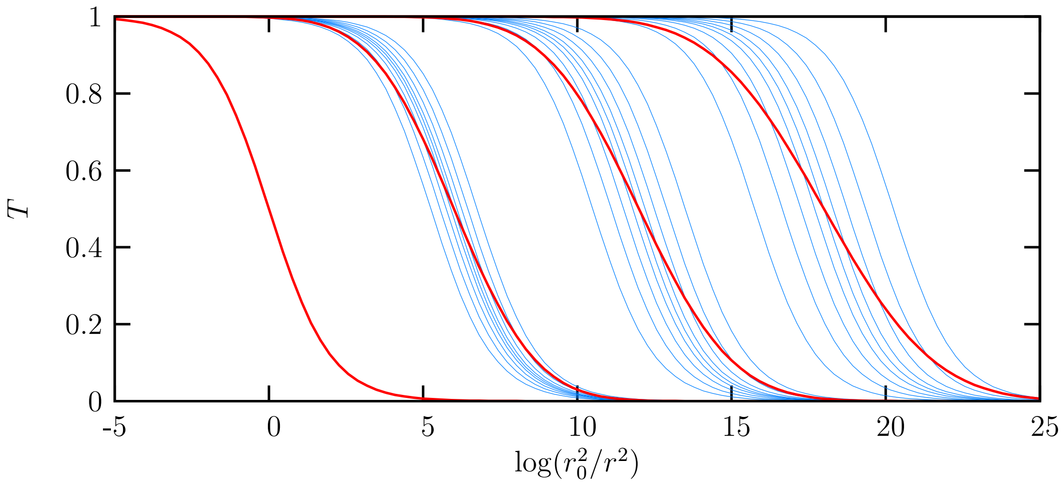


$$\langle T(r,Y)\rangle = \int d\rho_s\, T_{\text{event}}(\rho-\rho_s)\,\frac{1}{\sqrt{\pi}\sigma}\exp\left(-\frac{(\rho_s-\bar{\rho}_s)^2}{\sigma^2}\right)$$

with $\rho = \log(1/r^2)$, $\rho_s = \log(Q_s^2)$

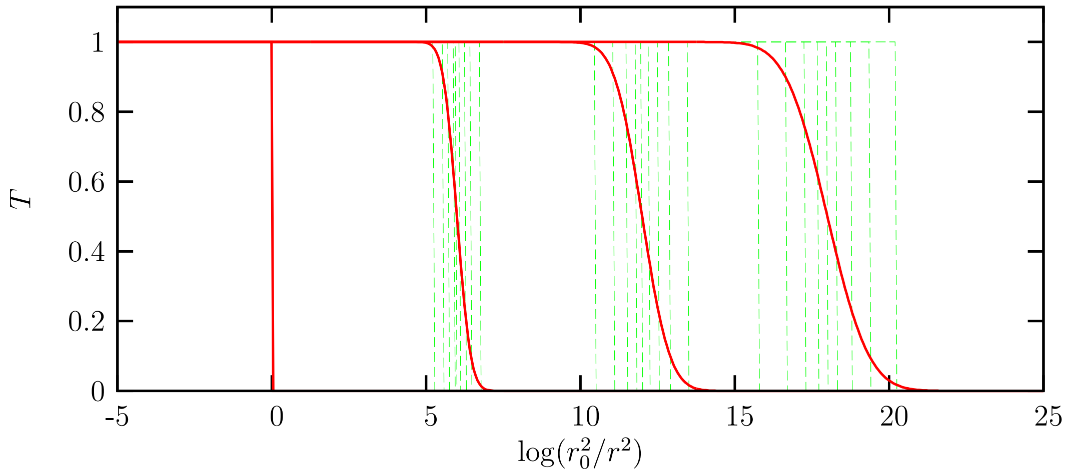
$$T_{\text{event}}(\rho-\rho_s) = \begin{cases} 1 & r > Q_s \\ (r^2Q_s^2)^\gamma & r < Q_s \end{cases}$$

High-energy behaviour



dispersion $\sim DY$

- Y not too large $\Rightarrow$ small dispersion $\Rightarrow \langle T \rangle \approx T_{\text{event}} \Rightarrow$ geometric scaling
- Y very high $\Rightarrow$ dominated by disperion *i.e.* $\langle T \rangle \approx T_{\text{sat}}$



NB.: $\langle T^2 \rangle = \langle T \rangle$

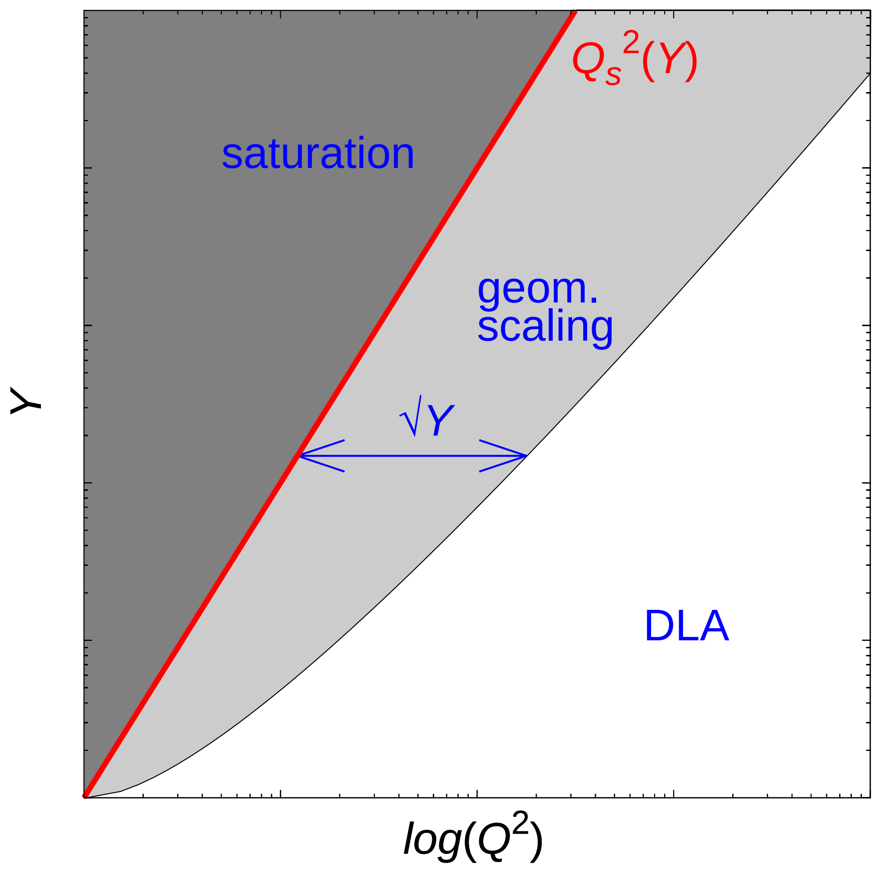
Strong noise limit

Intermediate energies	High energies
Mean field (BK)	Fluctuations
Geometric scaling $\langle T\rangle = f\left[\log(k^2/Q_s^2)\right]$ $\left\langle T^{(k)}\right\rangle = \langle T\rangle^k$	Diffusive scaling $\langle T\rangle = f\left[\log(k^2/Q_s^2)/\sqrt{DY}\right]$ $\left\langle T^{(k)}\right\rangle = \langle T\rangle$

At high-energy, amplitudes are dominated by black spots *i.e.* rare fluctuations at saturation

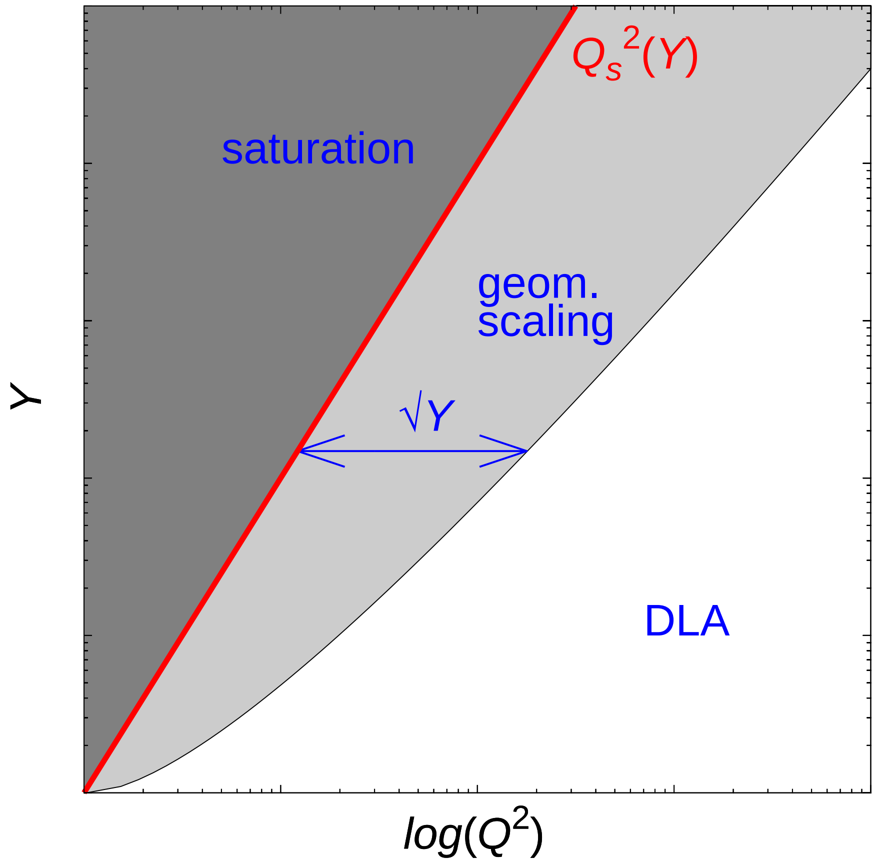
- true for strong fluctuations
- asymptotically true in general

saturation:

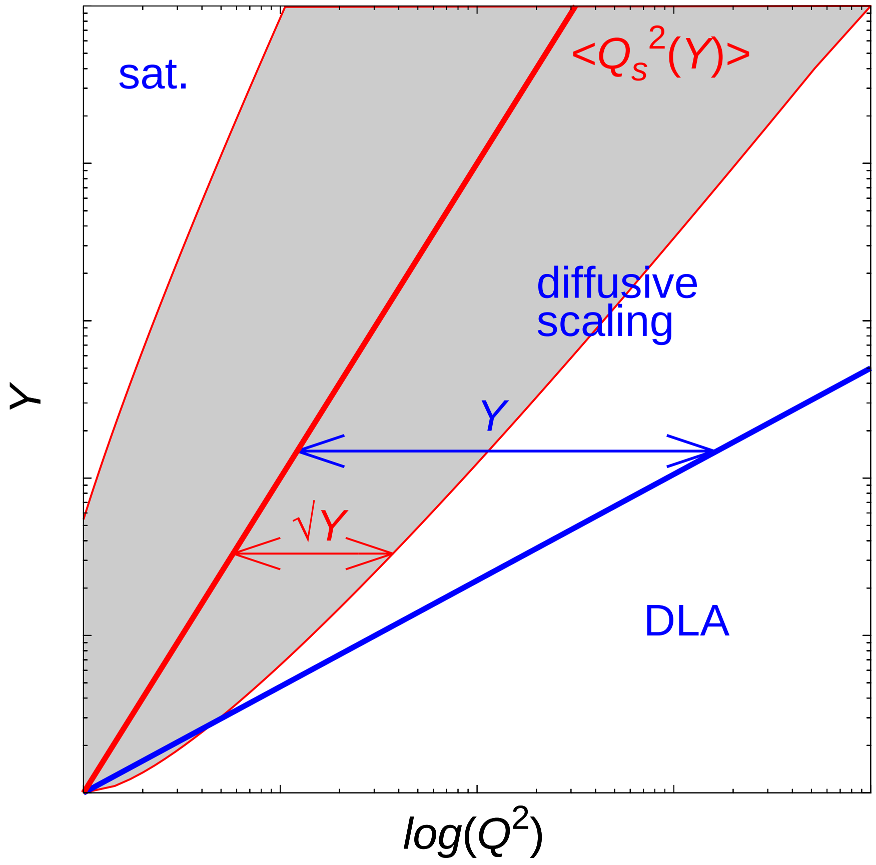


Phase space & scaling

saturation:



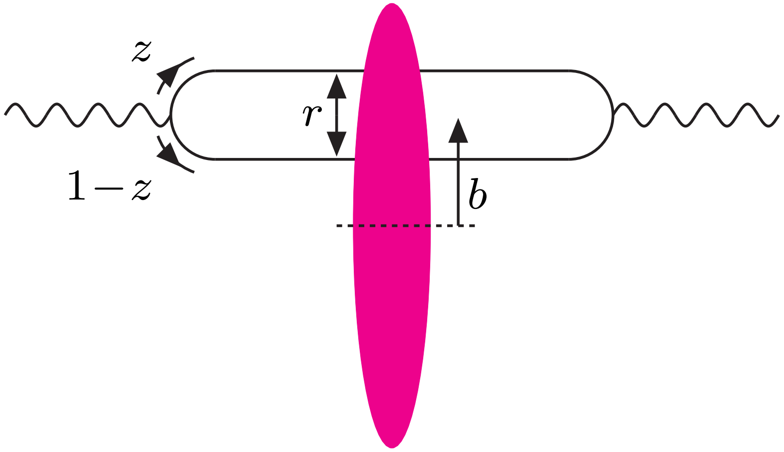
saturation+fluctuations:



Application: description of F_2^p data.

Factorisation formula: (see C. Marquet's talk for more details)

$$\frac{\sigma_{L,T}^{\gamma^* p}}{d^2 b} = \int d^2 r \int_0^1 dz \left|\Psi_{L,T}(z, r; Q^2)\right|^2 T(\mathbf{r}, \mathbf{b}; Y)$$



- $\Psi \equiv$ photon wavefunction $\gamma^* \to q\bar{q}$: QED process
- $T \equiv$ scattering amplitude from high-energy QCD.

$$\int d^2 b\, T(r, b, Y) = 2\pi R_p^2\, T(r; Y) \qquad F_2 = \frac{Q^2}{4\pi\alpha_e} \left[\sigma_L^{\gamma^* p} + \sigma_T^{\gamma^* p}\right]$$

Describing F_2

Saturation fit: [Iancu, Itakura, Munier]

$$\langle T(r,Y)\rangle = \begin{cases} (r^2Q_s^2)^{\gamma_c} e^{-\frac{2\log^2(rQ_s)}{CY}} & r < Q_s \\ 1 - e^{-a-b\log^2(rQ_s)} & r > Q_s \end{cases}$$

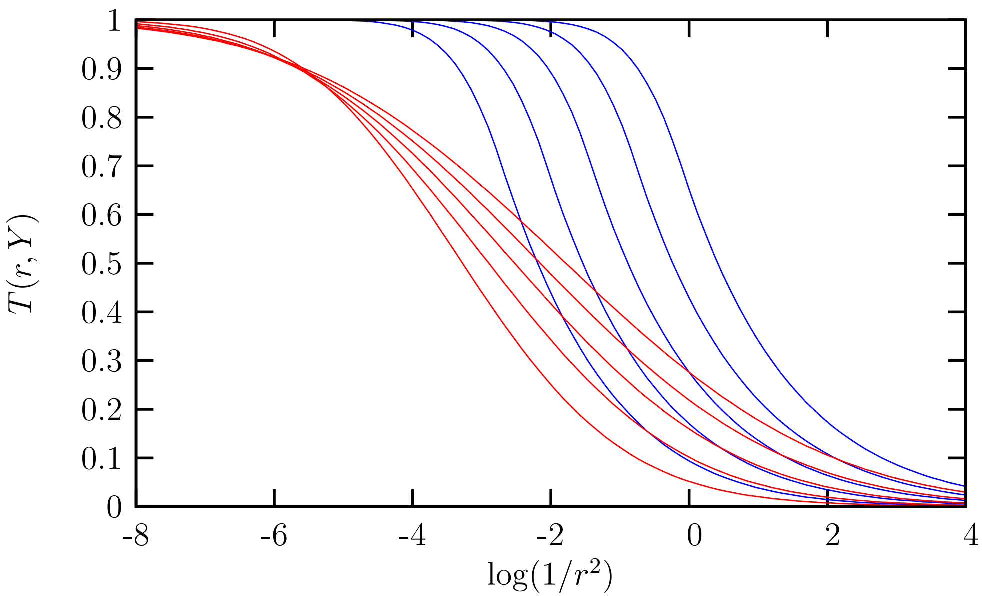
$$Q_s^2(Y) = \lambda Y,\ \rho_s = \log(Q_s^2)$$

Saturation+fluctuations fit: [in preparation]

$$\langle T(r,Y)\rangle = \int d\rho_s\, T(r,\rho_s) \frac{1}{\sqrt{\pi}\sigma} e^{-\frac{(\rho_s-\bar{\rho}_s)^2}{\sigma^2}}$$

$$T(r,\rho_s) = \begin{cases} r^2Q_s^2 & r < Q_s \\ 1 & r > Q_s \end{cases}$$

colour transparency

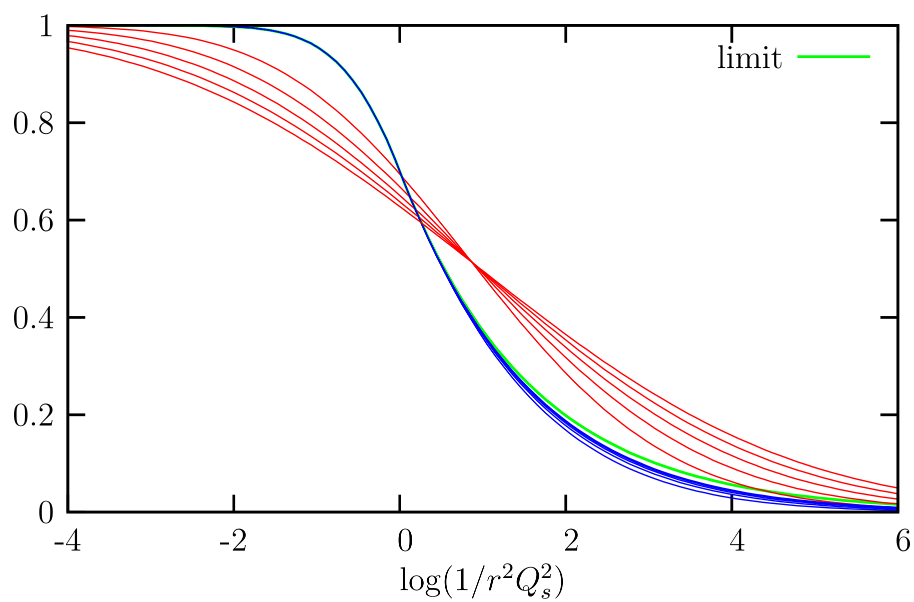


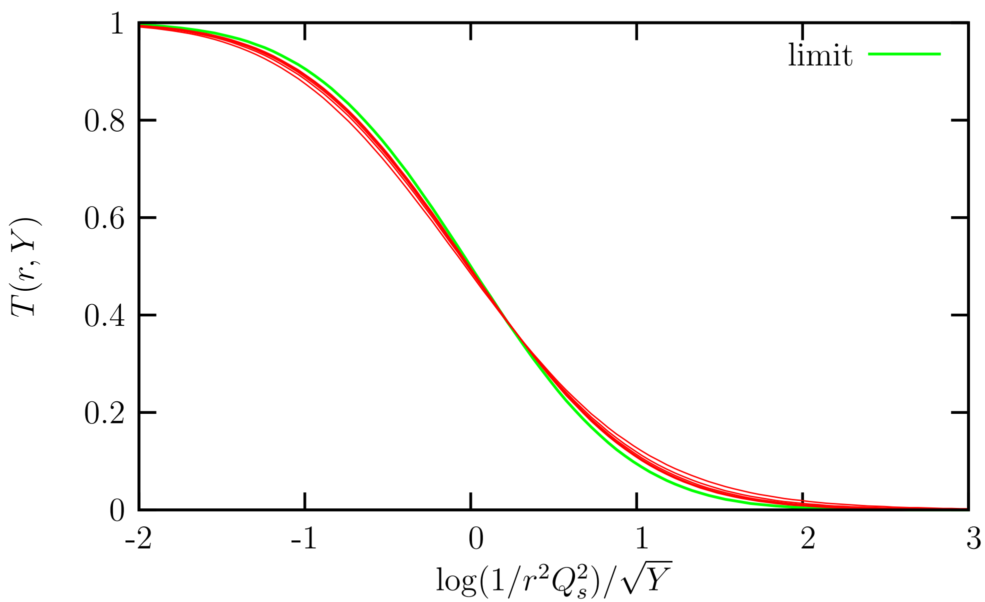
Saturation fit: [Iancu, Itakura, Munier]

$$\langle T(r,Y)\rangle = (r^2Q_s^2)^{\gamma_c} e^{-\frac{2\log^2(rQ_s)}{CY}} \qquad \to r^2Q_s^2$$

Saturation+fluctuations fit: [in preparation]

$$\langle T(r,Y)\rangle = \int d\rho_s\, T(r,\rho_s)\frac{1}{\sqrt{\pi}\sigma} e^{-\frac{(\rho_s-\bar\rho_s)^2}{\sigma^2}} \qquad \to \frac{1}{2}\text{erfc}\left(\frac{-\log(r^2\bar Q_s^2)}{\sqrt{2DY}}\right)$$

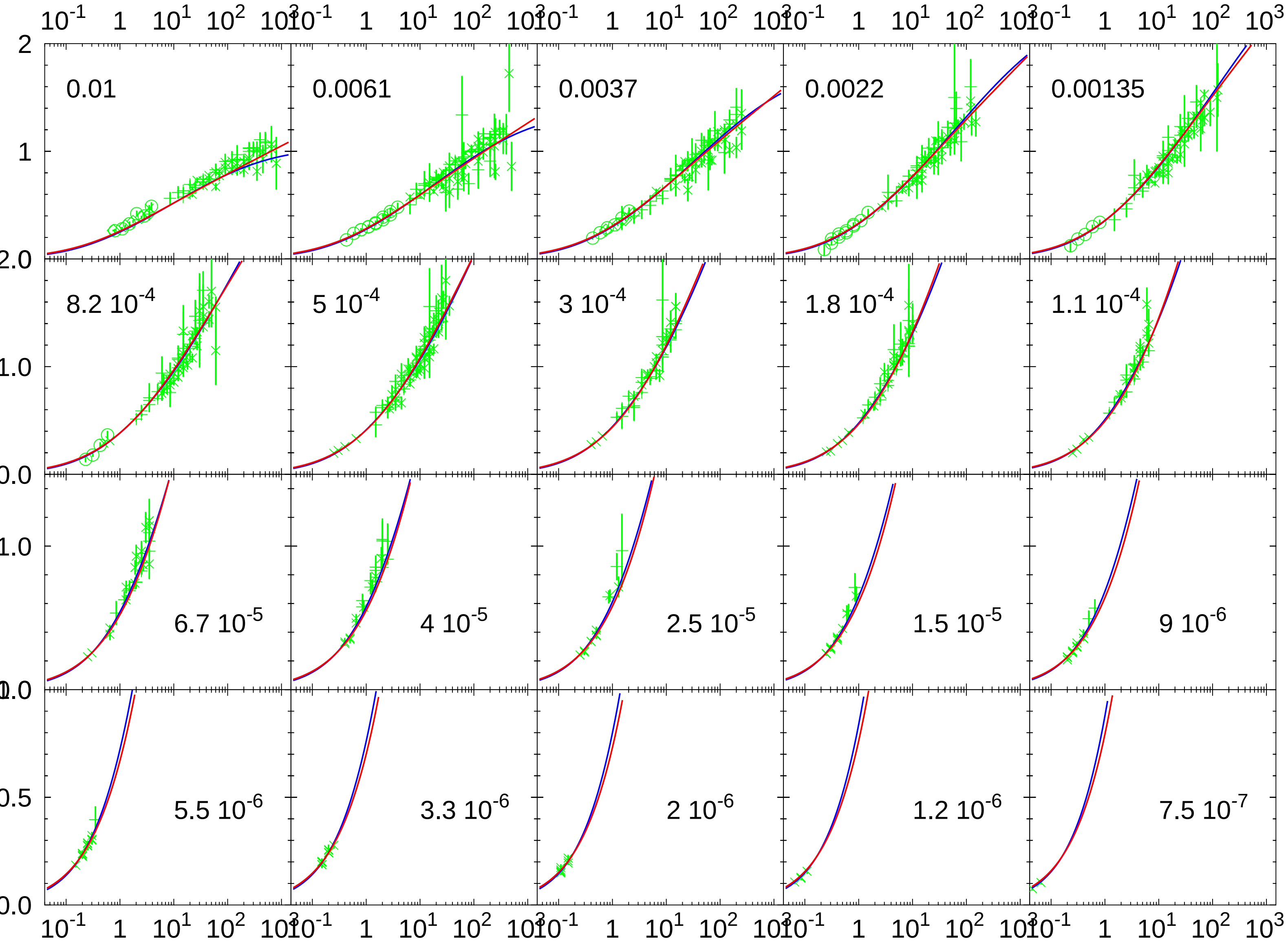




$\xrightarrow{Y\to\infty}$ Geometric scaling

$\xrightarrow{Y\to\infty}$ Diffusive scaling

Describing F_2



Both fits can describe the data for $x \leq 0.01$

- Effects of saturation
  - Evolution equations for high-energy QCD
    Balitsky, Balitsky-Kovchegov (dipole), JIMWLK (CGC)
  - Good knowledge of the asymptotic solutions
    Travelling waves $\rightarrow$ geometric scaling, saturation scale $\propto \exp(\bar{\alpha} v_c Y)$
- Effects of fluctuations
  - Known at large-N_c
  - analytical solutions: $\alpha_s \lll 1$
    numerical solutions: coherent with statistical-physics analog
  - Consequences on saturation (*e.g.* geometric scaling violations)
    black spots $\Rightarrow$ Diffusive scaling

Perspectives

- phenomenological tests:
  - do we observe geometric scaling at nonzero momentum transfer ?
  - predictions for LHC ? diffusive scaling at high-energy ?

- phenomenological tests:
  - do we observe geometric scaling at nonzero momentum transfer ?
  - predictions for LHC ? diffusive scaling at high-energy ?
- theoretical questions:
  - importance of geometric scaling violations
  - analytical predictions
    - speed, dispersion coefficients
    - pomeron loops, triple pomeron vertex
  - numerical simumations:
    - go beyond local-noise approximation
    - include impact parameter
  - beyond large-N_c